NAW Problem 29

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The Problem

Let n and h be natural numbers with $n > 0$ and A be a subset of $\{1, 2, ..., n+h\}$ with size n. Count the number of bijective maps $\pi : \{1, 2, ..., n\} \to A$ such that $k \leq \pi(k) \leq k + h$ for all $1 \leq k \leq n$.

Solution

Let $A = \{a_1, a_2, ..., a_n\}$ be a subset of $\{1, 2, 3, ..., n+h\}$, with $1 \leq a_1 < a_2 < ... < a_n \leq n+h$ and $(n > 0, h \geq 0)$. We are looking for permutations π of the elements of A with restrictions on permitted positions such that $k \leq \pi(k) \leq k + h$ for all $1 \leq k \leq n$. With this restrictions we can associate a (0,1)-matrix $B = [b_{ij}]$, where $b_{ij} = 1$, if and only if a_j is permitted in position i, meaning $0 \leq a_j - i \leq h$. We define S_B as the set of all permitted permutations, to be more precise

$$S_B = \{\pi | \prod_{i=1}^{n} b_{i\pi(i)} = 1\}$$

The number of elements of S_B can be calculated by

$$|S_B| = \sum_{\pi} \prod_{i=1}^{n} b_{i\pi(i)} = per(B)$$

where $per(B)$ is the permanent of B.

Example

Let $n = 4$, $h = 3$ and $A = \{2, 3, 5, 6\}$. We can easily see that in this case we have

$$B = \begin{pmatrix} 1\,1\,0\,0 \\ 1\,1\,1\,0 \\ 0\,1\,1\,1 \\ 0\,0\,1\,1 \end{pmatrix}$$

and $per(B) = 5$, so there are 5 permitted permutations. Being $(2, 3, 5, 6)$, $(3, 2, 5, 6)$, $(2, 3, 6, 5)$, $(3, 2, 6, 5)$ and $(2, 5, 3, 6)$.

Implementation

An implementation of the algorithm can be found on the website of the author: http://www.jaapspies.nl/mathfiles/problem29.c
For a given n and h this program calculates for all possible subsets A the number of allowed bijective maps.

Literature

[1] R.A Brualdi, H.J. Ryser, Combinatorial Matrix Theory, Cambridge University Press, 1991.
[2] H. Minc, Permanents, Reading, MA: Addison-Wesley, 1978.