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The problem

Introduction

Denote the fractional part of a positive real number x by $\{x\}$. Evaluate the following double integral:

$$\int_0^1 \int_0^1 \left\{\frac{x}{y}\right\} \left\{\frac{y}{x}\right\} dxdy$$

Solution

Let $z = \left\{\frac{x}{y}\right\} \left\{\frac{y}{x}\right\}$, then

$$\begin{aligned} z &= \left(\frac{x}{y} - \left\lfloor\frac{x}{y}\right\rfloor\right)\left(\frac{y}{x} - \left\lfloor\frac{y}{x}\right\rfloor\right) = 1 - \frac{x}{y}\left\lfloor\frac{y}{x}\right\rfloor - \frac{y}{x}\left\lfloor\frac{x}{y}\right\rfloor + \left\lfloor\frac{x}{y}\right\rfloor\left\lfloor\frac{y}{x}\right\rfloor \\ &= \begin{cases} 0 & \text{if} \quad x = y \\ 1 - \frac{x}{y}\lfloor\frac{y}{x}\rfloor & \text{if} \quad x < y \\ 1 - \frac{y}{x}\lfloor\frac{x}{y}\rfloor & \text{if} \quad x > y \end{cases} \end{aligned}$$

for $0 < x \leq 1$ and $0 < y \leq 1$.
By symmetry we have

$$I = \int_0^1 \int_0^1 z\, dxdy = 2\int_0^1 \int_0^y z\, dxdy = 2\int_0^1 \int_0^y (1 - \left\lfloor\frac{y}{x}\right\rfloor \frac{x}{y})dxdy$$

Now if $n \leq \frac{y}{x} < n+1$ we have $\lfloor\frac{y}{x}\rfloor = n$, $z = 1 - n\frac{x}{y}$ and $\frac{1}{n+1}y < x \leq \frac{1}{n}y$.
We define

$$I_n = \int_0^1 \int_{\frac{1}{n+1}y}^{\frac{1}{n}y} (1 - n\frac{x}{y})dxdy$$

We can easily check that

$$I_n = \frac{1}{4}(\frac{1}{n} - \frac{1}{n+1} - \frac{1}{(n+1)^2}) = \frac{1}{4n(n+1)^2}$$

and

$$I = 2\sum_{n=1}^{\infty} I_n = 2\sum_{n=1}^{\infty} \frac{1}{4n(n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{2n(n+1)^2} = 1 - \frac{\pi^2}{12}$$