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Jaap Spies

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The problem

Introduction

Let G be a finite group with n elements. Let c be the number of pairs $(g_1, g_2) \in G \times G$ such that $g_1 g_2 = g_2 g_1$. Show that either G is commutative or that $8c \leq 5n^2$. Show that if $8c = 5n^2$ then 8 divides n.

Solution

We need some elementary group theory and notation. Let $Z(g)$ be the centralizer of $g \in G$ and $K(g)$ the conjugacy class containing g. Z is the center of G.
We now have $|G| = n$, $c = \sum_{g \in G} |Z(g)|$ and

$$|K(g)| = [G : Z(g)] = \frac{|G|}{|Z(g)|} = \frac{n}{|Z(g)|}$$

We write the ratio

$$\begin{aligned} r &= \frac{c}{n^2} = \frac{\sum_{g \in G} |Z(g)|}{n^2} = \\ &= \frac{1}{n^2} \cdot \sum_{g \in G} \frac{n}{|K(g)|} = \\ &= \frac{1}{n} \cdot \sum_{g \in G} \frac{1}{|K(g)|} = \frac{k}{n} \end{aligned}$$

where k is the number of conjugacy classes.
We note that $r = 1$ if and only if G is commutative. So from now on let G be a non Abelian group.
We proof the following lemma:

The order of G/Z can not be a prime number.
Proof: As groups with order a prime are cyclic it is enough to proof that G/Z can not be cyclic. Suppose G/Z be cyclic generated by Zx. We get

$$G = Z \cup Zx \cup (Zx)^2 \cup (Zx)^3 \cup \cdots = Z \cup Zx \cup Zx^2 \cup Zx^3 \cup \cdots$$

and now arbitrary elements $g_1 = z_1 x^i$ and $g_2 = z_2 x^j$ clearly commute. This is a contradiction.
In order to maximise the number of conjugacy classes k we must maximise $|Z|$ the number of conjugacy classes with only one element. From the lemma it follows that $|G/Z| \geq 4$ and so $|Z| \leq \frac{1}{4}|G|$. The other conjugacy classes must have 2 or more elements. Hence

$$k \leq \frac{1}{4}|G| + \frac{1}{2} \cdot \frac{3}{4}|G| = \frac{5}{8}n$$

so that

$$r \leq \frac{5}{8}$$

and therefor

$$8c \leq 5n^2$$

.
If $r = \frac{5}{8}$ and hence $k = \frac{5}{8}n$ it is trivial that 8 is a divisor of n.