Problem A NAW 5/8 nr. 2

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The problem

Introduction

1. Find the largest number c such that all natural numbers n satisfy

$$n\sqrt{2} - \lfloor n\sqrt{2} \rfloor \geq \frac{c}{n}$$

2. For this c, find all natural numbers n such that $n\sqrt{2} - \lfloor n\sqrt{2} \rfloor = \frac{c}{n}$

Solution

Let $p = \lfloor n\sqrt{2} \rfloor$ and $q = n$, so we have to find the largest c for which

$$\sqrt{2} - \frac{p}{q} \geq \frac{c}{q^2} \tag{1}$$

holds for all natural numbers q.
We define a function $f : x \to x^2 - 2$. The equation $f(x) = 0$ has a solution $x = \sqrt{2}$. We note that $\frac{p}{q}$ is an approximation of $\sqrt{2}$ and that $1 \leq \frac{p}{q} < \sqrt{2}$.
Let M be the maximal value of $f'(x) = 2x$ in the interval $[1, \sqrt{2}]$, so $M = 2\sqrt{2}$.
Now $f(\frac{p}{q}) = (\frac{p}{q})^2 - 2 = \frac{p^2 - 2q^2}{q^2}$, hence $\left| f(\frac{p}{q}) - f(\sqrt{2}) \right| \leq \frac{1}{q^2}$.
By the mean-value theorem we get:

$$f(\frac{p}{q}) - f(\sqrt{2}) = f'(\xi)(\frac{p}{q} - \sqrt{2})$$

for some ξ in the interval $[1, \sqrt{2}]$.
Hence

$$\sqrt{2} - \frac{p}{q} = \left| \frac{p}{q} - \sqrt{2} \right| \geq \frac{1}{Mq^2} = \frac{\frac{1}{4}\sqrt{2}}{q^2}$$

Mutatis mutandi we have found $c = \frac{1}{4}\sqrt{2}$.
For this c there clearly is no n satisfying the equality of question 2.

Remark

According to [Hardy] the numbers $\sqrt{5}$ and $2\sqrt{2}$ play a crucial role in approximations of irrational numbers by rationals. For instance the Theorem: Any irrational $\xi \neq \frac{1}{2}(\sqrt{5}-1)$ has an infinity of rational approximations for which

$$\left|\frac{p}{q} - \xi\right| < \frac{1}{2q^2\sqrt{2}} = \frac{\frac{1}{4}\sqrt{2}}{q^2}$$

Interesting, isn't it?

Reference

[Hardy] Hardy, Wright, An Introduction to the Theory of Numbers, 5th edition, Oxford.