Problem B NAW 5/8 nr. 1

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The problem

Introduction

Given a non-degenerate tetrahedron (whose vertices do not all lie in the same plane), which conditions have to be satisfied in order that the altitudes intersect at one point?

Solution

Let T be such a tetrahedron with vertices A_0, A_1, A_2 and A_3 in a Euclidean space E. We define $\bar{a}_i = \vec{OA_i}$ and

$$\bar{e}_{ij} = \bar{a}_i - \bar{a}_j \tag{1}$$

for $i \neq j$. The vector $\bar{e}_{ij}$ is a direction vector of the edge A_iA_j.
The altitude h_i passing through A_i is determined by the following equations

$$\begin{aligned} \bar{e}_{jk} \cdot (\bar{a}_i - \bar{x}) &= 0 \\ \bar{e}_{jl} \cdot (\bar{a}_i - \bar{x}) &= 0 \\ \bar{e}_{kl} \cdot (\bar{a}_i - \bar{x}) &= 0 \end{aligned} \tag{2}$$

with $\{i, j, k, l\} = \{0, 1, 2, 3\}$.
Note that we need only two of them to determine h_i.
We now proof the following Lemma:
An altitude h_i intersects with an altitude h_j if and only if

$$\bar{e}_{kl} \cdot \bar{e}_{ij} = 0 \tag{3}$$

Proof: From (2), the altitude h_i is determined by the equations

$$\begin{aligned} \bar{e}_{jk} \cdot (\bar{a}_i - \bar{x}) &= 0 \\ \bar{e}_{kl} \cdot (\bar{a}_i - \bar{x}) &= 0 \end{aligned}$$

and the altitude h_j is determined by the equations

$$\begin{aligned} \bar{e}_{li} \cdot (\bar{a}_j - \bar{x}) &= 0 \\ \bar{e}_{kl} \cdot (\bar{a}_j - \bar{x}) &= 0 \end{aligned}$$

Let P be on h_i and h_j. Let $\bar{p}$ be the point vector of P. Then $\bar{e}_{kl} \cdot (\bar{a}_i - \bar{p}) = 0$ and $\bar{e}_{kl} \cdot (\bar{a}_j - \bar{p}) = 0$ and hence

$$\bar{e}_{kl} \cdot (\bar{a}_i - \bar{a}_j) = \bar{e}_{kl} \cdot \bar{e}_{ij} = 0$$

From (3) it follows that $\bar{e}_{kl} \cdot \bar{a}_i = \bar{e}_{kl} \cdot \bar{a}_j$ and so two of the four equations are equal. Three planes intersect in one point unless they are parallel to a line. This is clearly not the case since T is non-degenerate and the vectors $\bar{e}_{jk}$, $\bar{e}_{kl}$ and $\bar{e}_{il}$ are independent. So h_i and h_j must have a point in common.
For reasons of symmetry the same holds for the altitudes h_k and h_l.
Definition: A tetrahedron is called orthocentric if the altitudes intersect in one point.
Theorem: The following statements are equivalent:
$i)$ T is orthocentric.
$ii)$ All opposite edges are orthogonal.
Proof: $i) \Rightarrow ii)$. This follows immediately from the lemma.
$ii) \Rightarrow i)$. We now have

$$\bar{e}_{ij} \cdot \bar{e}_{kl} = \bar{e}_{ik} \cdot \bar{e}_{jl} = \bar{e}_{il} \cdot \bar{e}_{jk} = 0$$

So by the lemma, any two altitudes intersect. The four altitudes are not in the same plane, so there must be a common point.