Problem A NAW 5/8 nr. 1

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The problem

Introduction

Define the sequence $\{u_n\}$ by $u_1 = 1$, $u_{n+1} = 1 + (n/u_n)$. Prove or disprove that

$$u_n - 1 < \sqrt{n} \le u_n$$

Solution

By definition we have $u_1 = 1$, $u_{n+1} = 1 + (n/u_n)$ and hence $u_n(u_{n+1} - 1) = n$. So $\sqrt{n} = \sqrt{u_n(u_{n+1} - 1)}$ is the geometric mean of u_n and $u_{n+1} - 1$ and therefor

$$u_{n+1} - 1 \le \sqrt{n} \le u_n$$

for all $n \ge 1$.
Further we have for $n \ge 2$

$$u_n - 1 \le \sqrt{n-1} < \sqrt{n}$$

This inequality holds also for $n = 1$, so we proved

$$u_n - 1 < \sqrt{n} \le u_n$$

for all $n \ge 1$.