Problem C NAW 5/7 nr. 3 september 2006

Jaap Spies

October 2006

The problem

Introduction

Consider the triangle ABC inscribed in an ellipse. For given A the other vertices can be adjusted to maximize the circumference. Prove or disprove that this maximum circumference is independent of the position of A on the ellipse.

Solution

We show that the maximal circumference is independent of point A.
Let E and E' be ellipses with the same foci F_1 and F_2. Ellipse E' is inside E and sufficiently close to E. Let the points A, B, C and X be on E, such that the line segments AB, BC and CX are tangent to E'. In general the points A and X do not coincide, but if we shrink ellipse E' continously this will be eventually the case for $E' = E_0$. In this situation we have a triangle ABC inscribed in E with maximal circumference.
If we now move point A along ellipse E, we see that according to a theorem of Poncelet we always have a triangle inscribed in E and circumscribed around E_0 with by construction a maximal circumference. Chasles, Darboux and others proved that all this triangles have the same maximal circumference. See for example [1] Livre III, Chapitre III, part 176.
We follow the historical proof of Darboux, using 'infinitesimal' arguments. Let triangle ABC be defined as above. We have tangents AP and AQ with P and Q on ellipse E_0. Triangle ABC is a 'billiard triangle', meaning the tangents make equal angles tot the normal of E in point A. We move A to A' over an 'infinitesimal' distance and the corresponding points P and Q move to P' and Q'. Taking in account the properties of tangents we have

$$d(AP) = -AA'cos(A'AP) + PP'$$

and

$$d(AQ) = -AA'cos(A'AQ) - QQ'$$

and with the fact that the angles $A'AP$ and $A'AQ$ are supplementary, we get

$$d(AP + AQ) = PP' - QQ' = d(arc(PQ))$$

Hence the difference $D = (AP + AQ) - arc(PQ)$ is constant.

Doing this for all vertices of triangle ABC this leads to $3D = O - O'$, O being the circumference of ABC and O' the perimeter of the ellipse E_0.
Conclusion: The maximal circumference is independent of the position of A.
This problem can easily be generalized to an n-sided (convex) polygon inscribed in an ellipse for $n \geq 3$.

Remark

For a treatment independent of Poncelet's Theorem see George Lion, Variational Aspects of Poncelet's Theorem, Geometricae Dedicata 52, 105-118, 1994.

References

[1] Darboux, G: Principes de Géométrie analytique, Gauthier-Villars, Paris, 1917. Available in facsimile: http://gallica.bnf.fr