Problem B NAW 5/7 nr. 2

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The problem

Introduction

Imagine a flea circus consisting of n boxes in a row, numbered $1, 2, ..., n$. In each of the first m boxes there is one flea ($m \leq n$). Each flea can jump upwards or forwards to boxes with a maximal distance $d = n - m$. For all fleas all $d + 1$ jumps have the same probability.
The director of the circus has marked m boxes to be special targets. On his sign all m fleas jump simultaneously.
1. Calculate the probability that after the jump exactly m boxes are occupied.
2. Calculate the probability that all the m marked boxes are occupied.

Solution

Part 1

Let $d = n - m$. The jumps of the fleas corresponds to a bipartite graph G. We can associate a (0,1)-matrix B of size m by n with this graph. We have $b_{ij} = 1$ if and only if $i \leq j \leq i + d$. A matching M with cardinality t corresponds in the matrix B to a set of t ones with no two of the ones on the same line. The total number of jumps with exactly m boxes occupied is the number of matchings with $|M| = m$ is $per(B)$, the permanent of B. See [1], p. 44.
The asked probability is $\frac{per(B)}{(d+1)^m}$.

Part 2

Let A be the set of marked boxes, so $A = \{a_1, a_2, ..., a_m\}$ is a subset of $\{1, 2, 3, ..., n\}$, with $1 \leq a_1 < a_2 < ... < a_m \leq n$ and ($m > 0$, $m \leq n$). A succesful jump of the fleas can be associated with a permutation of the elements of A. We are looking for permutations π of the elements of A with restrictions on permitted positions such that $k \leq \pi(k) \leq k + d$ for all $1 \leq k \leq m$. With this restrictions we can associate a (0,1)-matrix $C = [c_{ij}]$, where $c_{ij} = 1$, if and only if a_j is permitted in position i, meaning $i \leq a_j \leq i + d$.
Compare Problem 29 from NAW 5/3 nr. 1 March 2002.
We define S_C as the set of all permitted permutations, to be more precise

$$S_C = \{\pi | \prod_{i=1}^{m} c_{i\pi(i)} = 1\} \tag{1}$$

The number of elements of S_C can be calculated by summing over all possible π

$$|S_C| = \sum_{\pi} \prod_{i=1}^{m} c_{i\pi(i)} = per(C) \tag{2}$$

where $per(C)$ is the permanent of C. See [2].
So the asked probability is $\frac{per(C)}{(d+1)^m}$.

References

[1] Brualdi, H.J. Ryser, Combinatorial Matrix Theory, Cambridge University Press, 1991.
[2] The Dancing School Problems: http://www.jaapspies.nl/mathfiles/problems.html