Problem C NAW 5/6 nr. 4 December 2005

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January 2006

The problem

Introduction

For a finite affine geometry there are a finite number of points and the axioms are as follows:

1. Given two distict points, there is exactly one line that includes both points.
2. The parallel postulate: Given a line L and a point P not on L, there exists exactly one line through P that is parallel to L.
3. There exists a set of four points, no three collinear.

We denote the set of points by $\mathbf{P}$, and the set of lines by $\mathbf{L}$. We define an automorphisme or collineation σ the usual way (a collineation keeps collinearity). Prove that there exist a point $P \in \mathbf{P}$ with $\sigma(P) = P$ or a line $L \in \mathbf{L}$ with $\sigma(L) = L$ or $\sigma(L) \cap L = \emptyset$.

Solution

Let π be a finite affine plane of order n. π can be canonically embedded in a projective plane $\bar{\pi}$ of order n by adding a line L_∞ and a point on every line L of π: $L \wedge L_\infty$, where parallel lines L and L' share the same point on L_∞.
$\bar{\pi}$ has n^2+n+1 points P_i and an equal number of lines L_i. Let $N = n^2+n+1$. We define an incidence matrix $A = (a_{ij})$ of order N:

$$a_{ij} = 1 \quad \text{if} \quad P_i \in L_j \quad \text{and} \quad a_{ij} = 0 \quad \text{if} \quad P_i \notin L_j$$

We see that

$$AA^T = A^T A = nI + J \tag{1}$$

with J a matrix with every entry 1.
A collineation σ of π can be extended to a collineation of $\bar{\pi}$, also indicated by σ. σ acts on the points P_i as a permutation P and as a permutation Q on the lines L_i. We write P and Q as $(0,1)$-matrices of order N with entries:

$$\begin{aligned} p_{ij} &= 1 \quad \text{if} \quad \sigma(P_i) = P_j \\ q_{ij} &= 1 \quad \text{if} \quad \sigma(L_i) = L_j \end{aligned}$$

and $p_{ij} = 0$, $q_{ij} = 0$ otherwise.

We now have

$$AQ = PA$$

and according to (1) we have

$$(det(A))^2 = det(nI + J) = (n+1)^2 n^{N-1} > 0$$

So

$$Q = A^{-1}PA$$

P and Q are similar as matrices, but also as permutations. Especially P and Q have the same number of cycles of length one, also called fixed "points".
$\sigma(L_\infty) = L_\infty$, so there must be at least one fixed point. If there are no fixed points on L_∞ there is a affine point P with $\sigma(P) = P$. If there is a fixed point on L_∞, say $L \wedge L_\infty$, then $\sigma(L) \parallel L$, meaning $\sigma(L) = L$ or $\sigma(L) \cap L = \emptyset$.