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The problem

Introduction.

1. Let G be a group and suppose that the maps $f, g : G \to G$ with $f(x) = x^3$ and $g(x) = x^5$ are both homomorphisms. Show that G is Abelian.
2. In the previous excercise, by which pairs (m, n) can $(3, 5)$ be replaced if we still want to be able to prove that G is Abelian.

Solution.

Part 1

Let $(ab)^5 = a^5b^5$ for all $a, b \in G$, then we easily see that $(ba)^4 = a^4b^4$. Now $(ab)^3 = a^3b^3$ for all $a, b \in G$ and hence $(ba)^2 = a^2b^2$. So $(a^2b^2)^2 = a^4b^4$ and $b^2a^2 = a^2b^2$. Hence in G squares commute.
Now $a^4b^4 = b^4a^4 = (ba)^4$ and so $b^3a^3 = (ab)^3 = a^3b^3$ and hence in G cubes commute. In the solution of Opgave 2003-4B from the UWC it is proved that in this case G is Abelian.

Part 2

We define $f_n(x) = x^n$ for $x \in G$. f_m and f_n are homomorphisms.
The case $m = 2$ $(m \leq n)$ is trivial because from $(ab)^2 = a^2b^2$ follows immediately $ba = ab$, etcetera.
From $(ab)^m = a^mb^m$ and $(ab)^n = a^nb^n$ $(m < n)$ follows $(ba)^{m-1} = a^{m-1}b^{m-1}$ and $(ba)^{n-1} = a^{n-1}b^{n-1}$.
If $k(m-1) = n-1$ and $m-1 = n-m$ we get $(a^{m-1}b^{m-1})^k = a^{n-1}b^{n-1}$. So $n = 2m-1$ and $k = 2$ and we may conclude that the $(m-1)$-th powers commute.
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See also

Vlastimil Dlab, A note on powers of a group, Acta Sci. Math. (Szeged) 25, 1964, pp. 177-178.