Problem A NAW 5/6 nr. 2 June 2005

Jaap Spies

July 2005

The problem

Introduction

A student association organises a large-scale dinner for 128 students. The chairs are numbered 1 through 128. The students are also assigned a number between 1 and 128. As the students come into the room one by one, they must sit at their assigned seat. However, 1 of the students is so drunk that he can't find his seat and takes an arbitrary one. Any sober student who comes in and finds his seat taken also takes an arbitrary one. The drunken student is one of the first 64 students. What is the probability that the last student gets to sit in the chair assigned to him?

Solution

We solve this problem for n students with $n > 1$. Without loss of generality we may assume that the first student is drunk, see below.
There are three possibilities: student 1 seats on seat 1 (we call this success, because student n will be seated on seat n), student 1 seats on seat n (failure) or student 1 seats on a remaining arbitrary seat k.
In the last case the next students with numbers less than k will be seated on their assigned seat. Student k will now act as a drunken student by taking an arbitrary free seat. So in a way student k becomes the new number '1' of a corresponding problem with $n - k + 1$ students. The choices are: Student '1' seats on seat 1 (success), he or she seats on the last seat (failure) or again the choice of an arbitrary free seat. This process is repeated until we have eventually two students, the 'first' and the last and there are only two choices, one leads to success, the other to failure.
In the end 'success' and 'failure' are completely symmetric in this story. In all possible stages of the process success and failure have the same probability, so the probability the last student will be seated on the last chair is 1/2.