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The problem

Introduction

We call a triangle integral if the sides of the triangle are integral. Consider the integral triangles with rational circumradius.

1. Prove that for any positive integral p there are only a finitely many integrals q such that there exists an integral triangle with circumradius equal to $\frac{p}{q}$.
2. Prove that for any positive integral q there exist infinitely many integral triangles with circumradius equal to $\frac{p}{q}$ for an integral p with $\gcd(p, q) = 1$.

Solution

Let triangle ABC have integral sides a, b and c with area A and circumradius R. There exists a relation between this quantities given by Heron's formulae

$$(4A)^2 = (a + b + c)(a + b - c)(a - b + c)(-a + b + c) \tag{1}$$

and

$$A = \frac{abc}{4R} \qquad \text{or equivalently} \qquad R = \frac{abc}{4A} \tag{2}$$

So there is an one-to-one relation between the set of all integral triangles with rational/integral area and the set of all integral triangles with rational circumradius. The set of all integral triangles with integral area is well studied as the Heronian triangles.

Part 1

The sides being integral implies the existance of a minimal circumradius R_{min}. For a given p there exist only finitely many q with $\frac{p}{q} > R_{min}$. So there are only finitely many q with $R = \frac{p}{q}$ the circumradius of an integral triangle.

Part 2

The case $q = 1$ is trivial, as it is a well known fact that there are infinitely many numbers p where p is the hypothenuse of a Pythagorean triangle. Scaling by two gives an integral triangle with circumradius $R = p$.
For the case $q > 1$ we use a parametric representation of the Heronian triangles as found in [1]

$$a = n(m^2 + k^2) \tag{3}$$
$$b = m(n^2 + k^2) \tag{4}$$
$$c = (m + n)(mn - k^2) \tag{5}$$
$$A = kmn(m + n)(mn - k^2) \tag{6}$$

For any integers m, n and k with $mn > k^2 > \frac{m^2 n}{(2m+n)}$, $\gcd(m, n, k) = 1$ and $m \geq n \geq 1$ we have one member of each simularity class of the Heronian triangles.
Using this and (2) we get

$$R = \frac{(m^2 + k^2)(n^2 + k^2)}{4k} \tag{7}$$

In our case we do not need the restriction to unique reduced Heronian triangles. For the problem at hand we only need the triangle inequalities $a+b > c$, $a+c > b$ and $b + c > a$, together with $mn - k^2 > 0$. As we can easily see this can be realised by $m > k$, $n > k$ and $k \geq 1$.
Let $k = q$ and $p = \frac{(m^2+q^2)(n^2+q^2)}{4}$. All we have to prove is the existance of infinitely many (m, n) such that 4 is a divisor of $(m^2 + q^2)(n^2 + q^2)$. If q is even than choose $n > q$ with $\gcd(n, q) = 2$ so $4|(n^2 + q^2)$, let $m > q$ be a positive integer with $\gcd(m, q) = 1$. If q is not even choose $n > q$ and $m > q$ both not even with $\gcd(n, q) = \gcd(m, q) = 1$, so $2|(n^2 + q^2)$ and $2|(m^2 + q^2)$.
The sums $m^2 + q^2$ and $n^2 + q^2$ or $(\frac{n}{2})^2 + (\frac{q}{2})^2$ are so called primitive sums of two squares, defined by $x^2 + y^2$ with $\gcd(x, y) = 1$. For a prime divisor of such a primitive sum $x^2 + y^2$ it is not possible to be a divisor of y. In all cases we easily verify that we have a Heronian triangle with circumradius $R = \frac{p}{q}$ with $\gcd(p, q) = 1$, so we have infinitely many of them.

Reference

[1] Buchholz, R. H., Perfect Pyramids, Bull. Austral. Math. Soc. 45, nr 3, 1992.