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The problem

Introduction

Calculate

$$\sum_{n=1}^{\infty} \frac{1}{\sum_{i=1}^{n} i^2}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{\sum_{i=1}^{n} i^3}$$

Solution

This kind of problems make me feel young. They remind me to the early sixtees and the lectures of Prof. Van der Blij.

Part 1

By a well known result we first write $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ and hence the first summand can be written as

$$\frac{6}{n(n+1)(2n+1)} = \frac{6}{n} + \frac{6}{n+1} - \frac{24}{2n+1}$$

Let

$$S_n = \sum_{k=1}^{n} \frac{1}{\sum_{i=1}^{k} i^2} = 6\sum_{k=1}^{n} \frac{1}{k} + 6\sum_{k=1}^{n} \frac{1}{k+1} - 24\sum_{k=1}^{n} \frac{1}{2k+1}$$

so using a result on harmonic numbers we get

$$S_n = 6H_n + 6(H_n - 1) - 24(H_{2n+1} - \frac{1}{2}H_n - 1) = 18 - 24(H_{2n+1} - H_n)$$

H_n being the n-th harmonic number. We know $H_n = \ln n + \Delta_n$ with $\lim_{n\to\infty} \Delta_n = \gamma$, Euler's constant. Now with $H_{2n+1} - H_n = \ln(2n+1) - \ln n - \Delta_{2n+1} + \Delta_n$ we can easily see that $\lim_{n\to\infty}(H_{2n+1} - H_n) = \ln 2$ and therefor the first answer is $18 - 24\ln 2$.

Part 2

First we write $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$ and hence the second summand can be written as

$$\frac{4}{n^2(n+1)^2} = \frac{4}{(k+1)^2} + \frac{4}{k^2} - \frac{8}{k} + \frac{8}{k+1})$$

Let

$$S_n = \sum_{k=1}^{n} \frac{1}{\sum_{i=1}^{k} i^3} = \sum_{k=1}^{n} (\frac{4}{(k+1)^2} + \frac{4}{k^2} - \frac{8}{k} + \frac{8}{k+1})$$

so

$$S_n = 4\sum_{k=1}^{n} \frac{1}{(k+1)^2} + 4\sum_{k=1}^{n} \frac{1}{k^2} - 8\sum_{k=1}^{n} \frac{1}{k} + 8\sum_{k=1}^{n} \frac{1}{k+1}$$

and

$$S_n = 4(\sum_{k=1}^{n} \frac{1}{k^2} - 1) + 4\sum_{k=1}^{n} \frac{1}{k^2} - 8H_n + 8(H_n - 1) = 8\sum_{k=1}^{n} \frac{1}{k^2} - 12$$

So

$$\lim_{n\to\infty} S_n = 8 \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k^2} - 12 = 8 \cdot \frac{1}{6}\pi^2 - 12 = \frac{4}{3}\pi^2 - 12$$