NAW Problem 26

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March 2002

Abstract

Does there exist a triangle with sides of integral lengths such that its area is equal to the square of the length of one of its sides?
The answer is no.

The problem

Introduction.

Does there exist a triangle with sides of integral lengths such that its area is equal to the square of the length of one of its sides?

Solution 1.

We can scale to integral sides easily, so suppose we have a triangle with rational sides a, b and c with area A. Then the famous Heron formula gives

$$\begin{aligned} (4A)^2 &= (a+b+c)(a+b-c)(a-b+c)(-a+b+c) \\ &= (x+c)(x-c)(y+c)(-y+c) \\ &= (x^2-c^2)(-y^2+c^2) \end{aligned}$$

with $x = a+b$ and $y = a-b$.
Without loss of generality we may state that $A = 1$ and $c = 1$. So we have

$$(x^2-1)(y^2-1) = -16$$

Does this equation have rational solutions?

We make substitutions $X = x$ and $Y = y(x^2 - 1) - x^2$. Rearanging we get

$$(Y + X^2)^2 - (X^2 - 1)^2 = -16(X^2 - 1)$$

Substitution of $X = 1/U - Y$ and solving for U means solving a quartic with discriminant $-2Y^3 - 18Y^2 + 34Y + 306$. So we are looking for a rational solution of

$$Z^2 = -2Y^3 - 18Y^2 + 34Y + 306$$

$Y = -1$ and $Z = 16$ represents a solution, so we can write

$$v^2 = 2u^3 - 12u^2 - 64u + 256$$

where $v = Z$ and $u = -Y - 1$.
This equation of an elliptic curve can be transformed into a Weierstrass equation (N.B.: new meaning of x and y):

$$y^2 - 4xy + 64y = x^3 - 16x^2$$

Which can be reduced to minimal form

$$y_1^2 + x_1y_1 + y_1 = x_1^3 - x_1^2 - x_1$$

This means that we have the elliptic curve (17 A 4 [1,-1,1-1,0] 0 4) from the appropriate Cremona table. This curve has rank zero, so in the torsion group we find all rational solutions: (0,0), (0,-1) and (1,-1). It is easily verified that this result gives no solutions to our original problem.

Solution 2.

Suppose a triangle ABC exists with integral sides a, b and c with basis c, area c^2 and height $CD = 2c$. Let $BD = d$, then $d^2 = a^2 - 4c^2$ and $d = \sqrt{a^2 - 4c^2}$. We consider the case that ABC is obtuse (The acute case is left as an excercise).

$$\begin{aligned} b^2 &= (c + d)^2 + 4c^2 \\ &= (c + \sqrt{a^2 - 4c^2})^2 + 4c^2 \\ &= a^2 + c^2 + 2c\sqrt{a^2 - 4c^2} \end{aligned}$$

Here a, b and c are integral, so also $d = \sqrt{a^2 - 4c^2}$ must be an integer and therefore the triangles BDC and ADC are Pythagorean.
A well known result states that a Pythagorean triangle can be parametrized. We leave out some of the details. For BDC we have $a = BC = u^2 + v^2$, $BD = u^2 - v^2$ and $CD = 2uv$, with integers u, v and $u > v$. In triangle ADC

we have $AD = s^2 - t^2$ and $CD = 2st$, with integers s, t and $s > t$. While $AB = c = uv$ we have

$$\begin{aligned} s^2 - t^2 &= u^2 - v^2 + uv \\ st &= uv \end{aligned}$$

Dividing the lefthand side of the first equation by st and the righthand side by uv we get

$$\frac{s}{t} - \frac{t}{s} = \frac{u}{v} - \frac{v}{u} + 1$$

Substitution of $y = \frac{s}{t}$ and $x = \frac{u}{v}$ while $st = uv$ gives

$$y - \frac{1}{y} = x - \frac{1}{x} + 1$$

By multiplying with xy we get

$$x^2y - xy^2 + xy + x - y = 0$$

This cubic can eventually be transformed into a Weierstrass equation of an elliptic curve by Nagell's algorithm. At first we tried this by hand, but the famous Apecs lib for MapleV from Ian Connell did it within seconds. The command $Gcub(0, 1, -1, 0, 0, 1, 0, 1, -1, 0, 0, 0)$; returned among others:

present curve is, $A17 = [1, -1, 1, -1, 0]$

Meaning that we have an elliptic curve of rank zero while the order of the torsion group equals 4. So the members of the torsion group (1,-1), (0,0) and (0,-1) are the only rational solutions. Backsubstitution gives no rational solutions for x and y. We end up with a contradiction.

Solution 3.

We can scale to integral sides easily, so suppose we have a triangle with rational sides a, b and c with area A. Then the famous Heron formula gives

$$\begin{aligned} (4A)^2 &= (a + b + c)(a + b - c)(a - b + c)(-a + b + c) \\ &= (x + c)(x - c)(y + c)(-y + c) \\ &= (x^2 - c^2)(-y^2 + c^2) \end{aligned}$$

with $x = a + b$ and $y = a - b$.
Without loss of generality we may state that $A = 1$ and $c = 1$. So we have

$$(x^2 - 1)(y^2 - 1) = -16$$

Does this equation have rational solutions?
Making the substitutions $U = x$ and $V = y(x^2 - 1) - x^2$ and rearanging we get

$$(V + U^2)^2 - (U^2 - 1)^2 = -16(U^2 - 1)$$

and so

$$2U^2V + 18U^2 + V^2 - 17 = 0$$

with solution $U = 1$ and $V = -1$.
This cubic can eventually be transformed into a Weierstrass equation of an elliptic curve by Nagell's algorithm. At first we tried this by hand, but the Apecs package for MapleV from Ian Connell did it within split seconds. The command $Gcub(0, 2, 0, 0, 18, 0, 1, 0, 0, -17, 1, -1)$; returned among other information:

$$\text{present curve is, } A17 = [1, -1, 1, -1, 0]$$

Meaning that we have a well known elliptic curve of rank zero while the order of the torsion group equals 4. So the members of the torsion group (1,-1), (0,0) and (0,-1) are the only rational solutions. It is easily verified that this result gives no solutions to our original problem.

Solution 4.

A Heron triangle is a triangle with sides of integral length and integral area. According to K.R.S Sastry [1] every Heron triangle with sides a, b and c can be parametrized as follows:
Let λ be a rational number such that $0 < \lambda \leq 2$.

$$(a, b, c) = (2(m^2 + \lambda^2 n^2), (2 + \lambda)(m^2 - 2\lambda n^2), \lambda(m^2 + 4n^2)),$$

m, n being relative prime natural numbers such that $m > \sqrt{2\lambda}.n$. For the sides we have $a \geq c$ and $gcd(a, b, c) = 1$. The area of this triangle $\Delta = 2\lambda(2 + \lambda)mn(m^2 - 2\lambda n^2)$.
As one can see Δ is always a multiple of b. So looking for a solution of our problem we have to consider $\Delta = b^2$, so

$$2\lambda mn = (2 + \lambda)(m^2 - 2\lambda n^2)$$

Dividing this equation by n^2 gives

$$2\lambda \frac{m}{n} = (2 + \lambda)((\frac{m}{n})^2 - 2\lambda)$$

Making the substitutions $U = \lambda$ and $V = \frac{m}{n}$ we get

$$UV^2 - 2U^2 - 2UV + 2V^2 - 4U = 0$$

with obvious solution $U = 0$ and $V = 0$.
This cubic can eventually be transformed into a Weierstrass equation of an elliptic curve by Nagell's algorithm. The Apecs package for MapleV from Ian Connell did this within split seconds.
The command $Gcub(0, 0, 1, 0, -2, -2, 2, -4, 0, 0)$; returned among other information:

$$\text{present curve is, } A17 = [1, -1, 1, -1, 0]$$

Meaning that we have a well known elliptic curve of rank zero while the order of the torsion group equals 4. So the members of the torsion group (1,-1), (0,0) and (0,-1) are the only rational solutions. It is easily verified that this result gives no solutions to our original problem.

[1] K.R.S Sastry, Heron Triangles: A Gergonne-Cevian-and-Median Perspective, Forum Geometricorum Volume 1 (2001) 17-24

Solution 5.

We can scale to integral sides easily, so suppose we have a triangle ABC with rational sides a, b and c with rational area Δ and $\angle ACB = \theta$. Since the area $\Delta = \frac{1}{2}ab\sin\theta$ is rational, $\sin\theta$ must be a rational number.
According to the law of cosines, $c^2 = a^2 + b^2 - 2ab\cos\theta$, so also $\cos\theta$ must be rational. Rational points on the unit circle can be parametrized as follows:

$$(\cos\theta, \sin\theta) = (\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2})$$

where the point (-1,0) or $\theta = \pi$ is excluded.
Without loss of generality we may state that $\Delta = 1$ and $c = 1$. So we have

$$\begin{aligned} 1 &= \frac{1}{2}ab\sin\theta \\ 1 &= a^2 + b^2 - 2ab\cos\theta \end{aligned}$$

and therefore $ab = \frac{2}{\sin\theta}$. This results in

$$a^2 + b^2 = 1 + \frac{2(1-t^2)}{t}$$

Let's try to investigate on this last expression. In triangle ABC we have height $CD = 2$ and let $AD = x$. We can treat the obtuse and acute case in one formula (the other case is trivial)

$$\begin{aligned} a^2 + b^2 &= (x-1)^2 + 4 + x^2 + 4 \\ &= (1-x)^2 + 4 + x^2 + 4 \\ &= 2x^2 - 2x + 9 \end{aligned}$$

so $2x^2t - 2xt + 9t = -2t^2 + t + 2$
It is clear that rational a and b give a rational x, but the converse is apparantly not true. See for instance $x = \frac{1}{2}$, meaning $a = b = \frac{1}{2}\sqrt{17}$.
Making the substitutions $U = x$ and $V = t$ and rearanging we get

$$2U^2V - 2UV + 2V^2 + 8V - 2 = 0$$

This cubic can be transformed into the minimal form of a Weierstrass equation of an elliptic curve by Nagell's algorithm.

$$Y^2 + YX + Y = X^3 - X^2 - 6X - 4$$

Meaning that we have the well known elliptic curve $(17A2[1, -1, 1, -6, -4]04)$ of rank zero while the order of the torsion group equals 4 from the Cremona table. So the members of the torsion group (3,-2), (-1,0) and (-5/4,1/8) are the only rational solutions. It is easily verified that this result gives no solutions to our original problem ($x = \frac{1}{2}$ and ($t = \frac{1}{4}$ or $t = -4$)).

Solution 6.

We can scale to integral sides easily, so suppose we have a triangle ABC with rational sides a, b and c with rational area Δ. Without loss of generality we may state that $\Delta = 1$ and $c = 1$. Let the height $CD = 2$ and $AD = x$. So, even without a picture we can see that

$$\begin{aligned} a^2 &= (1-x)^2 + 4 = (x-1)^2 + 4 \\ b^2 &= x^2 + 4 \end{aligned}$$

In our first attempt we eliminated x from this equations. This resulted in

$$(a^2 - b^2)^2 - 2(a^2 + b^2) + 17 = 0$$

Trying to solve this elegant equation we have to introduce a variable say x with $x^2 = b^2 - 4$. So we better take the shortcut.

As easily can be seen the rational solutions of our second equation can be parametrized with rational t by

$$x = \frac{4t}{1-t^2}, \quad b = \frac{2(t^2+1)}{1-t^2}$$

Substitution of x in the first equation gives

$$a^2 = \frac{5t^4 + 8t^3 + 6t^2 - 8t + 5}{(1-t^2)^2}$$

So we are looking for the rational solutions of the quartic

$$y^2 = 5t^4 + 8t^3 + 6t^2 - 8t + 5$$

with integer solution (1,4).
The command $Quar(5, 8, 6, -8, 5, 1, 4)$ from the Apecs package for MapleV returned among other information:

present curve is, $A17 = [1, -1, 1, -1, 0]$

Meaning that we have again the well known elliptic curve from the Cremona table (17 A 4 [1,-1,1,-1,0] 0 4) of rank zero while the order of the torsion group equals 4. So the members of the torsion group (1,-1), (0,0) and (0,-1) are the only rational solutions. It is easily verified that this result gives no solutions to our original problem.

Solution 7.

We can scale to integral sides easily, so suppose we have a triangle ABC with rational sides a, b and c with rational area Δ. Without loss of generality we may state that $\Delta = 1$ and $c = 1$. Let the height $CD = 2$ and $BD = x$. As we have seen before the rectangular triangle BDC has a corresponding Pythagorean triangle which can be parametrized using the integrals u and v. So we have

$$x = \frac{u^2 - v^2}{uv}, \quad CD = \frac{2uv}{uv} = 2 \quad \text{and} \quad a = BC = \frac{u^2 + v^2}{uv}.$$

and

$$\begin{aligned} b^2 &= 5 + 2x + x^2 \\ &= 5 + 2\frac{u^2 - v^2}{uv} + (\frac{u^2 - v^2}{uv})^2 \\ &= \frac{u^4 + 2u^3v + 3u^2v^2 - 2uv^3 + v^4}{u^2v^2} \end{aligned}$$

Dividing by v^4 and substituting $U = \frac{u}{v}$, we end up by searching rational solutions of

$$V^2 = U^4 + 2U^3 + 3U^2 - 2U + 1$$

with obvious solution (0,1).
The command $Quar(1, 2, 3, -2, 1, 0, 1)$ from the Apecs package for MapleV returned among other information:

$$\text{present curve is, } A17 = [1, -1, 1, -1, 0]$$

And we did it again! Meaning that we have once again the well known elliptic curve from the Cremona table (17 A 4 [1,-1,1,-1,0] 0 4) of rank zero while the order of the torsion group equals 4. So the members of the torsion group (1,-1), (0,0) and (0,-1) are the only rational solutions. It is easily verified that this result gives no solutions to our original problem.

Conclusion.

There is no such triangle.

With thanks to Ian Connell, John Cremona, James Milne and Dave Rusin.