NAW Problem 26

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The problem

Introduction.

Does there exist a triangle with sides of integral lengths such that its area is equal to the square of the length of one of its sides?

Solution 8.

A Heron triangle is a triangle with sides of integral length and integral area. We use a parametric representation of the Heronian triangles as found in [1]

$$\begin{aligned} a &= n(m^2+k^2) && (1)\\ b &= m(n^2+k^2) && (2)\\ c &= (m+n)(mn-k^2) && (3)\\ \Delta &= kmn(m+n)(mn-k^2) && (4) \end{aligned}$$

For any integers m, n and k with $mn > k^2 > \frac{m^2n}{(2m+n)}$, $\gcd(m,n,k) = 1$ and $m \geq n \geq 1$ we have one member of each simularity class of the Heronian triangles.
As one can see Δ is always a multiple of c. So looking for a solution of our problem we have to consider $\Delta = c^2$, so

$$kmn(m+n)(mn-k^2) = (m+n)^2(mn-k^2)^2$$

Or

$$kmn = (m+n)(mn-k^2)$$

Dividing this equation by n^3 gives

$$\frac{k}{n}\cdot\frac{m}{n} = (\frac{m}{n}+1)(\frac{m}{n}-(\frac{k}{n})^2)$$

Making the substitutions $U = \frac{m}{n}$ and $V = \frac{k}{n}$ we get

$$UV^2 - U^2 + UV + V^2 - U = 0$$

with obvious solution $U = 0$ and $V = 0$.
This cubic can eventually be transformed into a Weierstrass equation of an elliptic curve by Nagell's algorithm. The Apecs package for MapleV from Ian Connell did this with no pain.

The command $Gcub(0,0,1,0,-1,1,1,-1,0,0)$; returned among other information:

$$\text{present curve is, } A17 = [1,-1,1,-1,0]$$

Meaning that we have a well known elliptic curve of rank zero while the order of the torsion group equals 4. So the members of the torsion group (1,-1), (0,0) and (0,-1) are the only rational solutions. It is easily verified that this result gives no solutions to our original problem.

Conclusion.

There is no such triangle.

[1] Buchholz, R.H., Perfect Pyramids, Bull. Austral. Math. Soc. 45, nr 3, 1992.