The Dancing School Problems and the NAW Problem 29

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Abstract

We give results related to problem 29 of the NAW. There are connections to Mathematical Recreation and Graph Theory.

The Problem

Introduction.

The Dancing School Poblem:
Imagine a group of n ($n > 0$) girls ranging in integer length from m to $m+n-1$ cm and a corresponding group of $n+h$ boys ($h \geq 0$) with length ranging from m to $m+n+h-1$ cm. Clearly m is the minimal length of both boys and girls. The location is a dancing school. The teacher selects a group of n out of $n+h$ boys. A girl of length l can now choose a dancing partner out of this group of n boys, someone either of her own length or taller up to a maximum of $l+h$. How many 'matchings' are possible?
The proof of the equivalence of Problem 29 and the Dancing School Problem is left as an excercice.

A Solution

Let's return to the original problem of Lute Kamstra. Let $n > 0$ and $h \geq 0$ and let $A = \{a_1, a_2, ..., a_n\}$ be a subset of $\{1, 2, 3, ..., n+h\}$, with $1 \leq a_1 < a_2 < ... < a_n \leq n+h$. We are looking for permutations π of the elements of A with restrictions on permitted positions such that $k \leq \pi(k) \leq k+h$ for all k. With this restrictions we can associate a (0,1)-matrix $B = [b_{ij}]$, where $b_{ij} = 1$, if and only if a_j is permitted in position i, meaning $i \leq a_j \leq i+h$.
We define S_B as the set of all permitted permutations, to be more precise

$$S_B = \{\pi | \prod_{i=1}^{n} b_{i\pi(i)} = 1\} \tag{1}$$

The number of elements of S_B can be calculated by

$$|S_B| = \sum_{\pi} \prod_{i=1}^{n} b_{i\pi(i)} = per(B) \tag{2}$$

where $per(B)$ is the permanent of B.

For example, let $n = 4$, $h = 3$ and $A = \{2, 3, 5, 6\}$. We can easily see that in this case we have

$$B = \begin{pmatrix} 1\,1\,0\,0 \\ 1\,1\,1\,0 \\ 0\,1\,1\,1 \\ 0\,0\,1\,1 \end{pmatrix}$$

and $per(B) = 5$, so there are 5 permitted permutations.
Case closed? We know that in general calculating a permanent is a hard problem with algebraic complexity of order $n^2 2^n$. In some special cases there are more efficient algorithms.

Some Questions and Answers

Bipartite Graphs

Matrix B can be interpreted as an incidence matrix of a bipartite graph G with vertices in $X = \{1, 2, ..., n\}$ and $Y = A = \{a_1, a_2, ..., a_n\}$. An edge of G is a pair (i, a_j) with $b_{ij} = 1$. The edges of the example can be described as $E = \{(1, 2), (1, 3), (2, 2), (2, 3), (2, 5), (3, 3), (3, 5), (3, 6), (4, 5), (4, 6)\}$. A matching in G is a set of disjoint edges. A perfect matching is a matching containing n edges.
The number of perfect matchings is $per(B)$. Is it possible to calculate the number of perfect matchings with graph theory?

Rook Theory

This is a more algebraic approach of accounting for $|S_B|$. We interpret the matrix B as an $n \times n$ chess board. On squares with $b_{ij} = 1$ we may place a rook. Let $r_k(B)$ be the number of ways we can place k non-attacking rooks on the board (that is, choosing k squares in B no two are on the same line). This corresponds to a bipartite graph G, thus $r_k(B)$ is the number of matchings with k edges.
The rook polynomial $r(B, x)$ is defined as

$$r(B, x) = \sum_{k=0}^{n} r_k(B)x^k$$

So the number of perfect matchings is $r_n(B) = per(B)$.
Is there a simple way to calculate $r(B, x)$ from B? We don't think so, see also the next section.

Configuration Matrix

Let $m = \binom{n+h}{n} = \binom{n+h}{h}$ be the number of different subsets X_i of the set $X = \{1, 2, ..., n + h\}$. We define a (0,1) configuration matrix $C = [c_{ij}]$ with $i = 1, ..., m, j = 1, ..., n + h$ and $c_{ij} = 1$ if and only if $x_j \in X_i$.
The set A in the previous subsection is characterized by the row (0 1 1 0 1 1 0).
Is it possible to find a matrix B directly from a row of C?
Let A_k, $k = 1, 2, ..., m$ be a possible subset of X. In the row $[c_{kj}]$ let h be the number of entries with $c_{kj} = 0$, n the number of entries with $c_{kj} = 1$ and $A_k = \{j|c_{kj} = 1\} = \{a_1, a_2, ..., a_n\}$

We define matrix $B_k = [b_{ij}]$ of order n with $b_{ij} = 1$ if and only if $0 \leq a_j - i \leq h$. So the answer of Problem 29 for A_k is $per(B_k)$ for $k = 1, 2, ..., m$.

Related Poblems

Dancing School and Rooks

What if the girls take over power and put aside the teacher and they choose directly out of the set of $n + h$ boys (accepting the length restrictions)?
Clearly we can once again associate a bipartite graph G to this problem. The n-set X of girls and the $(n + h)$-set Y of boys provide the vertices. If a girl a can choose a boy b of appropriat length we have an edge $\{a, b\}$ of G.
The adjacency matrix A has a special form

$$A = \begin{pmatrix} O & B \\ B^T & O \end{pmatrix}$$

Here B is a (0,1)-matrix of size n by $n+h$ which specifies the adjacencies of the vertices of X and the vertices of Y. We have $b_{ij} = 1$ if and only if $i \leq j \leq i+h$.
A matching M with cardinality n corresponds in the matrix B to a set of n 1's with no two of the 1's on the same line. The total number of matchings with $|M| = n$ is $per(B)$.
It is clear that our problem can be translated into a Rooks Problem:
Find the number of all possible non-attacking placings of n rooks on a $n \times (n+h)$-chessboard, while placing a rook on the i-th row and the j-th column is restricted by the condition $i \leq j \leq i + h$.

Solutions?

Configuration Matrix

We tried to find a recursion from the configuration matrix of the previous section, the so called direct attack. We define the total number of matchings to be $f(n, h)$. We can rearrange the rows of C such that all rows with $c_{i,n+h} = 1$ are placed together. In this case we have $\pi(n) = n+h$, so the corresponding number of matchings is $f(n - 1, h)$. In all other rows we have $c_{i,n+h} = 0$, counting for $f(n, h - 1)$ matchings, but unfortunately also an extra amount where h comes in! So we can write

$$f(n, h) = f(n - 1, h) + f(n, h - 1) + x(n, h)$$

So far we are not very succesfull in finding expressions for $x(n, h)$.
In terms of the previous section we may also state

$$f(n, h) = \sum_{k=1}^{m} per(B_k)$$

We think this will not lead to any but trivial solutions, because the calculation of the permanent is a #P-complete problem. The most effective algorithm in general is Ryser's (see later) which is of order of complexity $O(n^2 2^n)$.

Rooks Polynomials

In theory it is possible to calculate the rook polynomial of arbitrary chessboards with the so called expansion theorem. Given a chessboard B, let $r_k(B)$ the number of of ways to put k non-attacking rooks on the board, and let

$$r(B,x) = \sum_{k=0}^{n} r_k(B)x^k$$

be the rook polynomial of board B and $(r_0(B), r_1(B), ..., r_n(B))$ the rook vector of B.

We mark a square on board B as special and denote B_s as the chessboard obtainted from B by deleting the corresponding row and column. B_d is the board obtained from B by deleting the special square. The ways of placing k non-attacking rooks can now be divided in two cases, those that have the rook in the special square and those that have not. In the fist case we have $r_{k-1}(B_s)$ possibilities and in the second $r_k(B_d)$. So we have the relation

$$r_k(B) = r_{k-1}(B_s) + r_k(B_d)$$

This corresponds to

$$r(B,x) = x\, r(B_s,x) + r(B_d,x)$$

This is the so called expansion formula.

Now we can find the rook polynomial of arbitrary boards by applying repeatedly the expansion formula. We think this is only feasible for small sizes, but maybe there are some hidden recursions.

The Permanent to the Rescue

As stated before we do have a solution to our problem: $per(B)$! B has a clear form compared to the previous section. So maybe there are solutions lying around.

There is one in Ryser's Algorithm: Let's try to translate Theorem 7.1.1. of [1] to our situation. Let $B = [b_{ij}]$ the $n \times (n+h)$ (0,1)-matrix with $b_{ij} = 1$ if and only if $i \le j \le i+h$. Let r be a number with $h \le r \le n+h-1$ and B_r an $n \times (n+h-r)$ sub-matrix of B. We define $\prod(B_r)$ to be the product of the row sums of B_r and $\sum\prod(B_r)$ the sum of all $\prod(B_r)$ taken over all choices of B_r. So

$$per(B) = \sum_{k=0}^{n-1} (-1)^k \binom{h+k}{k} \sum\prod(B_{h+k}) \qquad (3)$$

This is a solution, be it not very effective! But maybe we can do better in some cases.

The complements of ...

Intermezzo: Let A a (0,1)-matrix with m rows and n columns ($m \le n$). α is a k-subset of the m-set $\{1, 2, ..., m\}$ and β a k-subset of $\{1, 2, ..., n\}$. $A[\alpha, \beta]$ is the $k \times k$ submatrix of A determined by rows i with $i \in \alpha$ and columns j with $j \in \beta$.

The permanent $per(A[\alpha,\beta])$ is called a permanental k-minor of A. We define the sum over all possible α an β

$$p_k(A) = \sum_{\beta}\sum_{\alpha} per(A[\alpha,\beta])$$

We define $p_0(A) = 1$ and note that $p_m(A) = per(A)$. $p_k(A)$ counts for the number of k 1's with no two of the 1's on the same line, so $p_k(A) = r_k(A)$ of the rook vector of A.
According to theorem 7.2.1 of [1] we can evaluate the permanent of a (0,1)-matrix in terms of the permanental minors of the complementary matrix $J_{m,n} - A$, where $J_{m,n}$ is the m by n matrix with all entries 1.
Translated to our matrix B of this section we get

$$per(B) = \sum_{k=0}^{n} (-1)^k p_k(J_{n,n+h} - B) \frac{(n+h-k)!}{h!} \tag{4}$$

This is in particular interesting for $h \geq n-2$, in this case we can easily see that $p_k(J_{n,n+h} - A)$ is independent of h, meaning that $per(B) = f(n,h)$ is polynomial in h. For example we have:
$f(3,h) = h^3 + 3h$ $(h \geq 1)$,
$f(4,h) = h^4 - 2h^3 + 9h^2 - 8h + 6$ $(h \geq 2)$,
$f(5,h) = h^5 - 5h^4 + 25h^3 - 55h^2 + 80h - 46$ $(h \geq 3)$,
$f(6,h) = h^6 - 9h^5 + 60h^4 - 225h^3 + 555h^2 - 774h + 484$ $(h \geq 4)$,
$f(7,h) = h^7 - 14h^6 + 126h^5 - 700h^4 + 2625h^3 - 6342h^2 + 9072h - 5840$ $(h \geq 5)$,
We have polynomials up to $f(9,h)$.

The Free Dancing School

What if the girls choose directly out of the set of $n+h$ boys and don't accept the length restrictions? They may choose a boy of their own length or taller.
Here again B is a (0,1)-matrix of size n by $n+h$ which specifies the possible dancing pairs. We now have $b_{ij} = 1$ if and only if $i \leq j \leq n+h$. The number of matchings with cardinality n is $per(B)$.
Let $b_1, b_2, ..., b_m$ be integers with $0 \leq b_1 \leq b_2 \leq ... \leq b_m$. The m by b_m (0,1)-matrix $A = [a_{ij}]$ defined by $a_{ij} = 1$ if and only if $1 \leq j \leq b_i$, $(i = 1, 2, ..., m)$ is called a Ferrers matrix, denoted by $F(b_1, b_2, ..., b_m)$. According to [1] we can calculate the permanent with

$$per(F(b_1, b_2, ..., b_m)) = \prod_{i=1}^{m} (b_i - i + 1) \tag{5}$$

We can associate B with a Ferrers matrix $F(b_1, b_2, ..., b_n)$ with $b_i = h + i$. So

$$per(B) = \prod_{i=1}^{n} (h + i - i + 1) = (h+1)^n \tag{6}$$

A result we could also have found by direct counting, but we couldn't resist mentioning Ferrers matrices!

References

[1] R.A Brualdi, H.J. Ryser, Combinational Matrix Theory, Cambridge University Press.
[2] Nieuw Archief voor de Wiskunde (NAW), Problem Section: Problem 29.