# The permanent of a matrix of order $n$

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**Abstract**

A new algorithm or a new proof of a known result? Complexity is of the same order as Ryser's algorithm. Multiplication (and addition) with 1 and -1 is extremely easy. So this approach can probably be implemented very efficiently for (0,1) matrices.

# The Permanent

## Definitions

Let $A$ be a matrix of order $n$, the permanent of $A$ is defined by

$$per(A) = \sum_{\pi} a_{1\pi(1)} a_{2\pi(2)} ... a_{n\pi(n)} \tag{1}$$

while we sum over all $n!$ possible permutations $\pi$ of $1, 2, ..., n$.
We define a vector $\bar{x} = (x_1, x_2, ..., x_n)^T$ and a vector $\bar{y} = (y_1, y_2, ..., y_n)^T$. Let $\bar{y} = A\bar{x}$. We define a multivariate polynomial

$$\begin{aligned} P(x_1, x_2, ..., x_n) &= \prod_{i=1}^{n} y_i & (2) \\ &= (a_{11}x_1 + ... + a_{1n}x_n) \cdot \\ &\quad\ (a_{21}x_1 + ... + a_{2n}x_n) \cdot \\ &\quad\ \vdots \\ &\quad\ (a_{n1}x_1 + ... + a_{nn}x_n) & (3) \end{aligned}$$

All terms are of degree $n$.

In the expansion of (3) we are looking for the coefficient of the term with $x_1 \cdot x_2 \cdot ... \cdot x_n$. When we sum over all possible permutations, we get

$$\sum_\pi a_{1\pi(1)}x_{\pi(1)} \cdot a_{2\pi(2)}x_{\pi(2)} \cdot ... \cdot a_{n\pi(n)}x_{\pi(n)} =$$

$$= \left(\sum_\pi a_{1\pi(1)}a_{2\pi(2)}...a_{n\pi(n)}\right) \cdot x_1x_2...x_n$$

So $per(A)$ is the coefficient of the term with $x_1x_2...x_n$.
We define

$$Q(\bar{x}) = (\prod_{i=1}^{n} x_i) \cdot P(x_1, x_2, ..., x_n) \tag{4}$$

## A Theorem

Now we sum $Q(\bar{x})$ over all possible $\bar{x}$ with $x_i = \pm 1$.

$$\sum_{|\bar{x}|_\infty = 1} Q(\bar{x}) = \sum_{|x_i|=1} (x_1 \cdot x_2 \cdot ... \cdot x_n)\, P(x_1, x_2, ..., x_n)$$

$|\bar{x}|_\infty = 1$ meaning $|x_i| = 1$ for $i = 1, 2, ..., n$.
We can easily see that only the term with $x_1 \cdot x_2 \cdot ... \cdot x_n$ of $P(x_1, x_2, ..., x_n)$ is always counted positive. A term $t$ with factor $x_k$ missing in $P(x_1, ..., x_n)$, is counted once $t$ and once $-t$ so the overall result is 0. We have $2^n$ possible vectors $\bar{x}$ with $|\bar{x}|_\infty = 1$, so we have proved:

**Theorem 1** *The permanent of $A$ is*

$$per(A) = 2^{-n} \cdot \sum_{|\bar{x}|_\infty = 1} Q(\bar{x}) \tag{5}$$