Opgave C NAW 5/5 nr. 2 jun 2004

Jaap Spies

July 2004

The problem

Introduction

Let A be a ring and let $B \subset A$ be a subring. As a subgroup, B has finite index in A. Show that there exists a two-sided ideal I of A such that $I \subset B$ and I has finite index as a subgroup of A.

Solution

We have A and B as defined above. The index $[A : B] = k < \infty$ or with other words, the additive factor group A/B is a finite Abelian group build from cosets of type $x + B$.
Let $\mathcal{E}(G)$ be the ring of endomorphisms of the Abelian group G. We define a ring homomorphism f: $B \to \mathcal{E}(A/B)$: for $a \in B$ we define f: $a \mapsto \alpha$ with $(x + B)\alpha = xa + B$. Note that we use here the *right* function notation, avoiding the notion of anti-homomorphism (see [2]).
The kernel of f is $L = \{a \in B | Aa \subset B\}$, L is the largest left-ideal of A with $L \subset B$. The factor group B/L is isomorphic to a subgroup of $\mathcal{E}(A/B)$, so B/L is a finite Abelian group and since $(A/L)/(B/L) \cong A/B$ it follows that A/L is finite Abelian.
We now consider the ring homomorphism g: $A \to \mathcal{E}(A/L)$: for $b \in A$ we define g: $b \mapsto \beta$ with $\beta(x+L) = bx+L$. Its restriction to L, g_L: $L \to \mathcal{E}(A/L)$ has kernel $I = \{a \in L | aA \subset L\} = \{a \in B | Aa \subset B \wedge aA \subset B\}$.
I is the largest two-sided ideal of A with $I \subset B$. We have L/I finite and hence A/I is a finite Abelian group, so $[A : I] < \infty$.

References

[1] Marshall Hall, Jr. The Theory of Groups, Macmillan, New York, 1959.

[2] http://planetmath.org/encyclopedia/UnitalModule.html