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The problem

Introduction

For every integer $n > 2$ prove that

$$\sum_{j=1}^{n-1}\left(\frac{1}{n-j}\sum_{k=j}^{n-1}\frac{1}{k}\right) < \frac{\pi^2}{6}$$

Solution

Let

$$s_{n-1} = \sum_{j=1}^{n-1}\left(\frac{1}{n-j}\sum_{k=j}^{n-1}\frac{1}{k}\right) \tag{1}$$

We have $s_1 = 1$, $s_2 = 1\frac{1}{4}$, $s_3 = \frac{49}{36} = \frac{5}{4} + \frac{1}{9}$ and $s_4 = s_3 + \frac{1}{4^2}$.
We shall prove the following
Proposition

$$s_n = s_{n-1} + \frac{1}{n^2} \quad \text{for} \quad n > 1 \tag{2}$$

From this proposition follows:

$$s_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2} = \sum_{k=1}^{n}\frac{1}{k^2} \tag{3}$$

And we are finished, because $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k^2} = \frac{\pi^2}{6}$, we have $s_{n-1} < s_n < \frac{\pi^2}{6}$.
Note: this is true for $n \geq 2$.
Now we prove the proposition.

Let $A_{n-1} = (a_{ij}) = (\frac{1}{n-i} \cdot \frac{1}{j})$ with $1 \leq i \leq j \leq n-1$ and $B_n = (b_{ij}) = (\frac{1}{n+1-i} \cdot \frac{1}{j})$ with $1 \leq i \leq j \leq n$.
Then

$$s_{n-1} = \sum_{1 \leq i \leq j \leq n-1} a_{ij} \quad \text{and} \quad s_n = \sum_{1 \leq i \leq j \leq n} b_{ij}$$

Comparing a_{ij} with b_{ij} we see $b_{ij} = a_{i-1,j}$ for $2 \leq i \leq j \leq n-1$.
So

$$s_n = s_{n-1} - \sum_{i=1}^{n-1} a_{ii} + \sum_{j=1}^{n} b_{1j} + \sum_{i=1}^{n} b_{in} - b_{nn}$$

We can write $a_{ii} = \frac{1}{(n-i)i} = \frac{1}{n(n-i)} + \frac{1}{ni}$, so

$$\sum_{i=1}^{n-1} a_{ii} = \frac{1}{n} \sum_{i=1}^{n-1} \frac{1}{n-i} + \frac{1}{n} \sum_{i=1}^{n-1} \frac{1}{i} = \frac{1}{n} H_{n-1} + \frac{1}{n} H_{n-1} = \frac{2}{n} H_{n-1}$$

where H_{n-1} is the $(n-1)$-th harmonic number.
Further we know

$$\sum_{j=1}^{n} b_{1j} = \sum_{i=1}^{n} b_{in} = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{k} = \frac{1}{n} H_n$$

Now

$$s_n = s_{n-1} - \frac{2}{n} H_{n-1} + \frac{1}{n} H_n + \frac{1}{n} H_n - \frac{1}{n^2}$$

and hence

$$s_n = s_{n-1} + \frac{2}{n}(H_n - H_{n-1}) - \frac{1}{n^2} = s_{n-1} + \frac{2}{n} \cdot \frac{1}{n} - \frac{1}{n^2}$$

This concludes the proof of the proposition

$$s_n = s_{n-1} + \frac{1}{n^2} \tag{4}$$