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The problem

Introduction.

Let $(X_n)_{n\geq 1}$ be a sequence of independent and identically distributed random variables with $P\{X_n = 1\} = P\{X_n = -1\} = \frac{1}{2}$. Set $S_n = \sum_{k=1}^{n} X_k$. Calculate $P\{S_3 = 1 \vee S_6 = 2 \vee \cdots \vee S_{3n} = n \vee \cdots\}$.

Solution.

Let $\mathcal{P}(n) = P\{S_3 = 1 \vee S_6 = 2 \vee \cdots \vee S_{3n} = n\}$ and $A_n = \{1, 2, 3, ..., n\}$. We notice that

$$P\{S_{3k} = k\} = \frac{\binom{3k}{k}}{2^{3k}}$$

With the principle of inclusion/exclusion we get

$$\mathcal{P}(n) = P(n,1) - P(n,2) + \cdots + (-1)^{k-1}P(n,k) + \cdots (-1)^{n-1}P(n,n)$$

where

$$\begin{aligned} P(n,k) &= \sum_{\{i_1,i_2,\cdots,i_k\}\subset A_n} P\{S_{3i_1} = i_1 \vee S_{3i_2} = i_2 \vee \cdots \vee S_{3i_k} = i_k\} = \\ &= \sum_{i_1<i_2<\cdots<i_k\leq n} \frac{\binom{3i_1}{i_1}\binom{3i_2-3i_1}{i_2-i_1}\cdots\binom{3i_k-3i_{k-1}}{i_k-i_{k-1}}}{2^{3i_1}2^{3i_2-3i_1}\cdots 2^{3i_k-3i_{k-1}}} = \\ &= \sum_{i_1<i_2<\cdots<i_k\leq n} \frac{\binom{3i_1}{i_1}\binom{3i_2-3i_1}{i_2-i_1}\cdots\binom{3i_k-3i_{k-1}}{i_k-i_{k-1}}}{2^{3i_k}} \end{aligned}$$

We have to calculate $\lim_{n\to\infty} \mathcal{P}(n)$.

We see that

$$P(n+1,k) = \sum_{i_1<i_2<\cdots<i_k\le n+1} \frac{\binom{3i_1}{i_1}\binom{3i_2-3i_1}{i_2-i_1}\cdots\binom{3i_k-3i_{k-1}}{i_k-i_{k-1}}}{2^{3i_k}} =$$

$$= P(n,k) + \sum_{i_1<i_2<\cdots<i_{k-1}<i_k=n+1} \frac{\binom{3i_1}{i_1}\binom{3i_2-3i_1}{i_2-i_1}\cdots\binom{3n+3-3i_{k-1}}{n+1-i_{k-1}}}{2^{3n+3}}$$

and so

$$\mathcal{P}(n+1) = \mathcal{P}(n) + \mathcal{D}(n)$$

with

$$\mathcal{D}(n) = \sum_{k=1}^{n+1}(-1)^{k-1} \sum_{i_1<i_2<\cdots<i_{k-1}<i_k=n+1} \frac{\binom{3i_1}{i_1}\binom{3i_2-3i_1}{i_2-i_1}\cdots\binom{3n+3-3i_{k-1}}{n+1-i_{k-1}}}{2^{3n+3}}$$

We have $\mathcal{P}(2) = \mathcal{P}(1)+\mathcal{D}(1)$ and $\mathcal{P}(3) = \mathcal{P}(2)+\mathcal{D}(2) = \mathcal{P}(1)+\mathcal{D}(1)+\mathcal{D}(2)$, etcetera. Hence

$$\mathcal{P}(n) = \mathcal{P}(1) + \sum_{i=1}^{n-1}\mathcal{D}(i) = \frac{3}{8} + \sum_{i=1}^{n-1}\mathcal{D}(i)$$

Elementary counting gives the following results:

$$\begin{aligned}
\mathcal{D}(1) &= 6/64 = 0.093750\\
\mathcal{D}(2) &= 21/512 = 0.041016\\
\mathcal{D}(3) &= 90/4096 = 0.021973\\
\mathcal{D}(4) &= 429/32768 = 0.013092\\
\mathcal{D}(5) &= 2184/262144 = 0.008331\\
\mathcal{D}(6) &= 11628/2097152 = 0.005545\\
\mathcal{D}(7) &= 63954/16777216 = 0.003812\\
\mathcal{D}(8) &= 360525/134217728 = 0.002686\\
\mathcal{D}(9) &= 2072070/1073741824 = 0.001930\\
\mathcal{D}(10) &= 12096045/8589934592 = 0.001408
\end{aligned}$$

Total $\sum_{i=1}^{10}\mathcal{D}(i) = 1662515613/8589934592$, so

$$\mathcal{P}(11) = \frac{3}{8} + \sum_{i=1}^{10}\mathcal{D}(i) = 4883741085/8589934592 = 0.5685422901$$

We can do better: the sequence $a(n)_{n\geq 1} = 6, 21, 90, 429, 2184, 11628, \cdots$ can be written as:

$$a(n) = \frac{2}{3n+2}\binom{3n+3}{n+1}$$

and hence

$$\mathcal{D}(n) = \frac{a(n)}{2^{3n+3}}$$

Further we can write

$$\mathcal{P}(n) = \frac{3}{8} + \sum_{i=1}^{n-1} \mathcal{D}(i) = \frac{3}{8} + \sum_{i=1}^{n-1} \frac{\binom{3i+3}{i+1}}{(3i+2)2^{3i+2}} \tag{1}$$

The probability in question is

$$\lim_{n\to\infty} \mathcal{P}(n) = \frac{3}{8} + \sum_{i=1}^{\infty} \frac{\binom{3i+3}{i+1}}{(3i+2)2^{3i+2}} = 0.572949016875157727 69311\cdots$$

Conclusion.

The above calculations are based on a lemma:

$$\mathcal{D}(n) = \frac{2}{3n+2} P\{S_{3n+3} = n+1\} \tag{2}$$

This lemma can be proved with induction on n, proving

$$a(n) = 2^{3n+3} \cdot \mathcal{D}(n) == \frac{2}{3n+2}\binom{3n+3}{n+1} = \frac{3(3n+1)}{(2n+1)(n+1)}\binom{3n}{n}$$

The summand

$$\mathcal{D}(i) = \frac{\binom{3i+3}{i+1}}{(3i+2)2^{3i+2}}$$

is a hypergeometric term, but not 'Gosperable', so there is no closed form for $\mathcal{P}(n)$ in the sense of [1] Definition 8.1.1. See [1] and [2]. Maple 8 gives a ${}_3F_2$ hypergeometric form.

$$\lim_{n\to\infty} \mathcal{P}(n) = \frac{3}{8} + \sum_{i=1}^{\infty} \frac{\binom{3i+3}{i+1}}{(3i+2)2^{3i+2}} = \frac{3}{8} + \frac{3}{32} \cdot {}_3F_2(1, \frac{5}{3}, \frac{7}{3}; \frac{5}{2}, 3; \frac{27}{32})$$

which evaluates to $0.57294901687515772769311\cdots$.

References

[1] Petkovšek, Wilf and Zeilberger, $A = B$, A.K. Peters, Massachusetts, 1996.
[2] The Maple SumTools[Hypergeometric] library