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The problem

Introduction

For each non-negative integer n, let a_n be the number of digits in the decimal expansion of 2^n that are at least 5. Evaluate the sum $\sum_{n=0}^{\infty} \frac{a_n}{2^n}$.

Solution

Let $b(n)$ be the number of odd digits in the decimal expansion of 2^n. We can easily see that (change of notation) $a(n) = b(n+1)$, because a digit with value 5 of higher in 2^n generates an odd digit in the next generation 2^{n+1}. The sequence $b(n)$ is well known from Sloane's On-Line Encyclopedia of Integer Sequences as $A055254$. See [1] and [2].
We evaluate

$$\sum_{n=0}^{\infty} \frac{a(n)}{2^n} = \sum_{n=0}^{\infty} \frac{b(n+1)}{2^n}$$

We do not know a formula for $a(n)$ nor $b(n)$ other than an algorithm that can be implemented for instance in Maple.

```
A055254:=proc(n) local i, j, k, val;
val:= 2^n; j:=0; k:= floor(ln(val)/ln(10))+1;
for i from 1 to k do
   if (val mod 10) mod 2 = 1 then j:=j+1 fi;
   val:=floor(val/10);
od;
RETURN(j);
end:
```

When we calculate the sum of the first 200 terms we get:

$$\frac{357097343168664505675991576075813911671600665285065074511387}{1606938044258990275541962092341162602522202993782792835301376}$$

and approximately

0.22222222222222222222222222222222222222222222222222222222220362

With 1000 terms we even get more 2s in the decimal expansion, so there is circumstantial evidence enough to evaluate the sum in question to $\frac{2}{9}$.

This is not much of a proof. Is there a kind of CAS-equivalent of 'epsilontics':

For $\epsilon > 0$ there is a N_ϵ, such that $n > N_\epsilon$ implies $|\frac{2}{9} - \sum_{k=0}^{n} a(k)/2^k| < \epsilon$?

Not every ϵ can be represented in a computer system.

References

[1] N.J.A. Sloane, The On-Line Encyclopedia of Integer Sequences, Notices of the AMS, Vol. 50 nr 8 (September 2003), 912-915.

[2] http://www.research.att.com/ njas/sequences/

[3] http://www.maplesoft.com/