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The problem

Introduction.

Let $S(n)$ be the sum of the remainders on division of the natural number n by $2, 3, ..., n-1$. Show that

$$\lim_{n\to\infty} \frac{S(n)}{n^2}$$

exists and compute its value.

Solution.

We define the remainder in the division of n by k by $r_{n,k} = n - k \cdot [\frac{n}{k}]$, so from the definition of $S(n)$ it follows that

$$S(n) = \sum_{k=2}^{n-1} r_{n,k} = \sum_{k=1}^{n} r_{n,k} = \sum_{k=1}^{n}(n - k \cdot [\frac{n}{k}]) = n^2 - \sum_{k=1}^{n} k \cdot [\frac{n}{k}]$$

With $\sigma(k) = \sum_{d|k} d$, Theorem 324 and the proof of this Theorem taken from Hardy and Wright, An Introduction to the Theory of Numbers, 5th ed. p. 264-266, we get

$$\begin{aligned} S(n) &= n^2 - \sum_{x=1}^{n} \sum_{1\le y\le n/x} y = n^2 - \sum_{k=1}^{n} \sigma(k) \\ &= n^2 - (\frac{1}{12}\pi^2 n^2 + O(nlogn)) \end{aligned}$$

Hence

$$\lim_{n\to\infty} \frac{S(n)}{n^2} = 1 - \frac{1}{12}\pi^2$$