Opgave A NAW 5/4 nr. 3 sep 2003

Jaap Spies

September 2003

The problem

Introduction

Let $\{a_n\}_{n=0}^{\infty}$ be a non-decreasing sequence of real numbers such that $(n-1)a_n = na_{n-2}$ for $n = 1, 2, ...$ with initial value $a_0 = 2$. We have to calculate a_1.

Solution 1

We have $a_{2k-2} \leq a_{2k-1} \leq a_{2k}$ for $k \geq 2$.
The recursion $(n-1)a_n = na_{n-2}$ leads to the following results:
For $n = 2k - 1$

$$a_{2k-1} = \frac{2k-1}{2k-2} \cdot \frac{2k-3}{2k-4} \cdot ... \cdot \frac{3}{2} \cdot a_1$$

and for $n = 2k$

$$a_{2k} = \frac{2k}{2k-1} \cdot \frac{2k-2}{2k-3} \cdot ... \cdot \frac{2}{1} \cdot a_0 = 2k \cdot \frac{2k-2}{2k-1} \cdot \frac{2k-4}{2k-3} \cdot ... \cdot \frac{2}{3} \cdot \frac{1}{1} \cdot a_0$$

This can be written with double factorials as

$$a_{2k-1} = \frac{(2k-1)!!}{(2k-2)!!} \cdot a_1$$

and

$$a_{2k} = 2k \cdot \frac{(2k-2)!!}{(2k-1)!!} \cdot a_0$$

So

$$(2k-1) \cdot \frac{((2k-2)!!)^2}{((2k-1)!!)^2} \cdot a_0 \leq a_1 \leq 2k \cdot \frac{((2k-2)!!)^2}{((2k-1)!!)^2} \cdot a_0$$

As we can easily see

$$a_1 = \lim_{k\to\infty} 2k \cdot \frac{((2k-2)!!)^2}{((2k-1)!!)^2} \cdot a_0$$

From the properties of double factorials it follows that

$$(2k-2)!! = 2^{k-1} \cdot (k-1)! = 2^{k-1} \cdot \Gamma(k)$$

and

$$(2k-1)!! = \frac{2^k}{\sqrt{\pi}} \Gamma(\frac{1}{2} + k)$$

So with a well known limit we get

$$a_1 = \lim_{k\to\infty} \pi \cdot \frac{k(\Gamma(k))^2}{(\Gamma(\frac{1}{2}+k))^2} = \pi \cdot 1 = \pi$$

Solution 2

Without double factorials we can write

$$a_{2k-1} = \frac{(2k-1)!}{(2^{k-1} \cdot (k-1)!)^2} \cdot a_1$$

and

$$a_{2k} = \frac{2k(2^{k-1}(k-1)!)^2}{(2k-1)!} \cdot a_0$$

and hence

$$a_1 = \lim_{k\to\infty} \frac{2k(2^{k-1}(k-1)!)^4}{((2k-1)!)^2} \cdot a_0 = \lim_{k\to\infty} \frac{2k \cdot 2^{4(k-1)}}{(2k-1)^2 \binom{2k-2}{k-1}^2} \cdot a_0$$

Writing $a_0 = 2$ and $n = k-1$ we get with other well known limits

$$a_1 = \lim_{n\to\infty} \frac{4(n+1) \cdot 2^{4n}}{(2n+1)^2 \binom{2n}{n}^2} = \lim_{n\to\infty} \frac{4n^2+4n}{4n^2+4n+1} \cdot \frac{2^{4n}}{n\binom{2n}{n}^2} = 1 \cdot \pi = \pi$$