The Dancing School Problem

Jaap Spies

November 2003

A Solution

Part a

We number the the girls $1, 2, 3, ..., 20$ and the boys $1, 2, 3, ..., 30$ in increasing length. The maximal length difference is 7 [cm]. The poor boys 28, 29 and 30 are acting as wallflowers! The choice of the girls corresponds to a bipartite graph G. We can associate a (0,1)-matrix B of size 20 by 30 with this graph. We have $b_{ij} = 1$ if and only if $i \leq j \leq i + 7$. A matching M with cardinality 20 corresponds in the matrix B to a set of 20 ones with no two of the ones on the same line. The total number of 'classes' (matchings) with $|M| = 20$ is $per(B)$, de permanent van B:

$$per(B) = 16709568237540$$

Part b

We generalize. Let the maximal length difference be h. Let $A = \{a_1, a_2, ..., a_m\}$ be a subset of $\{1, 2, 3, ..., n\}$, with $1 \leq a_1 < a_2 < ... < a_m \leq n$ and ($m > 0$, $m \leq n$). We are looking for permutations π of the elements of A with restrictions on permitted positions such that $k \leq \pi(k) \leq k + h$ for all $1 \leq k \leq m$. With this restrictions we can associate a (0,1)-matrix $B = [b_{ij}]$, where $b_{ij} = 1$, if and only if a_j is permitted in position i, meaning $i \leq a_j \leq i + h$.
We define S_B as the set of all permitted permutations, to be more precise

$$S_B = \{\pi | \prod_{i=1}^{m} b_{i\pi(i)} = 1\} \tag{1}$$

The number of elements of S_B can be calculated by summing over all possible π

$$|S_B| = \sum_{\pi} \prod_{i=1}^{m} b_{i\pi(i)} = per(B) \tag{2}$$

where $per(B)$ is the permanent of B.
For example, let $m = 4$, $h = 3$ and $A = \{2, 3, 5, 6\}$. We can easily see that in this case we have

$$B = \begin{pmatrix} 1\,1\,0\,0 \\ 1\,1\,1\,0 \\ 0\,1\,1\,1 \\ 0\,0\,1\,1 \end{pmatrix}$$

and $per(B) = 5$, so there are 5 permitted permutations.

[1] R.A Brualdi, H.J. Ryser, Combinatorial Matrix Theory, Cambridge University Press, 1991.