

SCIENCE IS WHAT WE UNDERSTAND WELL ENOUGH TO EXPLAIN TO A COMPUTER. ART IS EVERYTHING ELSE WE DO. DURING THE PAST SEVERAL YEARS AN IMPORTANT PART OF MATHEMATICS HAS BEEN TRANSFORMED FROM AN ART TO A SCIENCE: NO LONGER DO WE NEED TO GET A BRILLIANT INSIGHT IN ORDER TO EVALUATE SUMS OF BINOMIAL COEFFICIENTS, AND MANY SIMILAR FORMULAS THAT ARISE FREQUENTLY IN PRACTICE; WE CAN NOW FOLLOW A MECHANICAL PROCEDURE AND DISCOVER THE ANSWERS QUITE SYSTEMATICALLY

DONALD E. KNUTH

DEAR JAAP, GOOD LUCK WITH YOUR PROBLEM-SOLVING! PROBLEM-SOLVERS ARE AN ENDANGERED SPECIES, AND IT IS NICE THAT PEOPLE LIKE YOU KEEP THE FLAME ALIVE. BEST WISHES

DORON ZEILBERGER

MY MOTTO FOR LIFE: "DER JAAP IST BRILLANT, ABER FAUL", WHICH TRANSLATES INTO: "JAAP IS BRILLIANT, BUT LAZY".

HANS FREUNDENTHAL

TO MANY LAYMEN, MATHEMATICIANS APPEAR TO BE PROBLEM SOLVERS, PEOPLE WHO DO "HARD SUMS". EVEN INSIDE THE PROFESSION WE CLASSIFY OURSELVES AS EITHER THEORISTS OR PROBLEM SOLVERS. MATHEMATICS IS KEPT ALIVE, MUCH MORE THAN BY THE ACTIVITIES OF EITHER CLASS, BY THE APPEARANCE OF A SUCCESSION OF UNSOLVED PROBLEMS, BOTH FROM WITHIN MATHEMATICS ITSELF AND FROM THE INCREASING NUMBER OF DISCIPLINES WHERE IT IS APPLIED. MATHEMATICS OFTEN OWES MORE TO THOSE WHO ASK QUESTIONS THAN TO THOSE WHO ANSWER THEM.

RICHARD K. GUY

JAAP SPIES

A BIT OF MATH

THE JOY OF PROBLEM SOLVING

SPIES PUBLISHERS

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SPU

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*Dedicated to all friends of Mathematics
and Good Thinking*

Introduction

Solving Math Problems

In the first epigraph I cite Donald E. Knuth: “Science is what we understand well enough to explain to a computer. Art is everything else we do.”

With this statement in mind, I came up with the subtitle of this book: *The Art of Problem Solving*. ‘Art’ is a simple word with many meanings. I don’t feel very comfortable with the idea of doing ‘art’, or being an artist. Problem solving is work — sometimes hard work, but most of the time a pleasant task. And in the end, there is sometimes a reward.

The joy of discovery — to know something you didn’t know before. And above all, it is learning in a pleasant way. I remember the feeling when I could write QED at the end of a proof in *Elementary Geometry*.

In the early sixties I enjoyed the ‘Practica’ alongside the lectures on Linear Algebra by Hans Freudenthal and on Analysis by Fred van der Blij. Making exercises, doing homework, and getting feedback from student assistants. In my third year at Utrecht University, I became an assistant myself.

After a ‘kandidaatsexamen’ in Mathematics, Physics and Astronomy, I pursued a Master’s in Pure Mathematics and the Foundations of Mathematics. I especially enjoyed the lectures of Dirk van Dalen on the Philosophy of Mathematics.

One of the textbooks I like most is Graham, Knuth and Patashnik: *Concrete Mathematics*.¹ The book contains more than 500 exercises — from ‘warmups’, ‘basics’, and ‘exam problems’ to ‘research problems’. Answers and extra information can be found in an appendix. The answers to the research problems are not complete! The margin notes are amusing. Problem solving — doing exercises — is an essential part of mathematical education.

You better read this introduction on my website www.jaapspies.nl, unless you are reading a pdf with link information.

See also Knuth: *The Art of Computer Programming*

You may have noticed. The subtitle on the cover differs from the one inside. The flag doesn’t cover the load!

¹ Ronald Graham, Donald Knuth, and Oren Patashnik. *Concrete mathematics : A Foundation for Computer Science*. Addison-Wesley, Reading, Mass, 1994. ISBN 0201142368

The importance of a Problem Section in mathematical journals should not be underestimated. Read ‘Opinion 19’ by Doron Zeilberger, where he warned *SIAM Review* in 1997 against discontinuing its Problem Section. He noted: “Problem sections turned many young people into mathematicians.” His first ‘publication’ was a solved problem in *SIAM Review*. “Both the problem and my solution were real gems”, he said. ...

Not only young people benefit from the existence of problem sections. Near the end of my professional career, I found my way back to mathematics — starting in December 2001 with a simply formulated problem: “Does there exist a triangle with sides of integral length such that its area is equal to the square of the length of one of its sides?”

I found this problem as Problem 26 in the Problem Section of the *Nieuw Archief voor Wiskunde* (NAW).

The NAW and its Problem Section

The NAW is a publication of the Dutch Royal Mathematical Society, the Koninklijk Wiskundig Genootschap (KWG). The society publishes the *Nieuw Archief voor Wiskunde* (NAW), a beautiful quarterly for all its members, featuring a renowned Problem Section.

The Problem Section has a long history. Problems and solutions have been published since the early days of the Society in 1778, and for several decades they constituted the most substantial part of the material in the Proceedings of the Society. The KWG is the world’s oldest national mathematical society.

My Solution of Problem 26

In the newsgroup sci.math, I discovered an article by Dave Rusin, linking the existence of certain triangles with rational sides to elliptic curves. From there, I found the Mathematical Atlas 14H52: Elliptic Curves. A real treasure trove! Learning from the well-written lecture notes of J.S. Milne on elliptic curves (1996), discovering the Maple package APECS from Ian Connell, and the EC tables of John Cremona, I came up with a solution. See



Figure 1: Doron Zeilberger: Opinion 19



Figure 2: NAW December 2001

<https://www.nieuwarchief.nl>



Figure 3: Mathematical Atlas 14H52 (Wayback Machine)

Chapter 30, Solution 1. There are no such triangles.

This was the start of finding a series of solutions. The latest was Solution 8 in July 2005, based on a parametric representation of Heronian triangles found in my solution of Problem 2005-1/C (Chapter 14).

You can find the published solution by Frits Beukers in the NAW 5/3 nr. 3, and my collection of solutions can be found on my website. And of course, here in this book as Chapter 30.

Next Problem

The next problem that surprised me was Problem 29 of the NAW. An erroneous solution had appeared in NAW 5/3 nr. 3, and the problem was declared open again in NAW 5/3 nr. 4. The editor of the Problem Section, Robbert Fokkink, challenged me to tackle this unsolved problem. I found a solution at the beginning of January 2003. Interestingly, Problem 29 originated from work related to a paper by Lute Kamstra: *Juggling Polynomials*, CWI Report PNA-R0113, July 2001.

For myself, I translated and extended the problem into a Dancing School Problem: "How to match boys and girls in a dancing class under certain height restrictions." My story of the Dancing Schools includes the solution of Problem 29, but also links it with certain types of Rook Placement Problems. There is a SAGE program to generate polynomial solutions to a certain class of problems. See also Chapter 32. From the Dancing School Problem originated the sequences A079908–A079928 from the OEIS.

My solution to Problem 29 is in terms of the permanent of $(0,1)$ -matrices. So I became interested in algorithms for permanents. Playing with small examples and doing some counting, I found an alternative to the famous Ryser's algorithm. See Chapter 33. I implemented my algorithm in a C/C++ program, which was used to contribute to Neil Sloane's On-Line Encyclopedia of Integer Sequences (OEIS). See for instance A087982, A088672, and A089476.

Problem 29 reappeared in disguise as Part 2 of Problem 2006-2B in NAW 5/7 nr. 2. It was included without my consent. No



Figure 4: NAW September 2002

See my website:

<https://www.jaapspies.nl/mathfiles>



Figure 5: NAW December 2002

<https://oeis.org>

solutions were submitted, which was an absolute waste of a nice problem!

There is an article in NAW 5/7 nr. 4 December 2006: Dancing School Problems, Permanent Solutions of Problem 29. In a sidenote of this paper, my formula for the permanent of a square matrix appeared for the first time. In the Dutch Wikipedia on permanents, it is known as the 'Formule van Spies'.



Figure 6: Dancing School Problems NAW Dec 2006

UWC/Problems

The Problem Section of the NAW was discontinued and merged with the UWC, the University Math Competition, open to Belgian and Dutch math students. Starting with NAW 5/4 nr.1, the Problem Section and the UWC became the section Problemen/UWC. Initially in Dutch, but later—at my suggestion—the problems were formulated in English, as were the solutions. Students can earn points for their solutions. Others may submit their solutions *hors concours*.

In NAW 5/5 nr. 3 there was no UWC/Problems section, due to a misunderstanding between the editorial board and the editors of the section. There was a change of editors beginning with NAW 5/5 nr. 4. Note the difference: “Opgave” was replaced by “Problem”.

The UWC has changed back into a general Problem Section, open to everyone.

My latest submission was a partial solution to Problem 2020-1/C. You will not find it here in this book.



Figure 7: Archive of Problems and Solutions since 2005

Global Contents

In Part I, selected problems from the NAW are presented almost exactly as I submitted them between 2003 and 2009.

In Part II, the Special Problems: Problem 26 and Problem 29 of the old Problem Section of the NAW.

In Part III, Permanent Questions from 2003 until now.

Personal Remarks

I would like to thank all the editors of the Problem Section. I especially want to thank Robbert Fokkink, who was until recently Editor-in-Chief of the *Nieuw Archief voor Wiskunde*. Without him, my way back to mathematics would not have existed.

I wrote this book for myself—just to see on paper what problem solving has brought me. I really enjoyed the work and the insights I gained through doing it.

My problem-solving activities brought me into contact with some outstanding mathematicians from around the world: Richard Brualdi, Edwin Clarke, Ian Connell, John Cremona, Richard Guy, Neil Sloane, Doron Zeilberger —to name a few.

Part I

Problems from the NAW

1

*Opgave A NAW 5/4 nr. 3, September 2003**The problem**Introduction*

Let $\{a_n\}_{n=0}^{\infty}$ be a non-decreasing sequence of real numbers such that
 $(n-1)a_n = na_{n-2}$ for $n = 1, 2, \dots$ with initial value $a_0 = 2$. We
 have to calculate a_1 .

Solution 1

We have $a_{2k-2} \leq a_{2k-1} \leq a_{2k}$ for $k \geq 2$.

The recursion $(n-1)a_n = na_{n-2}$ leads to the following results:

For $n = 2k - 1$

$$a_{2k-1} = \frac{2k-1}{2k-2} \cdot \frac{2k-3}{2k-4} \cdot \dots \cdot \frac{3}{2} \cdot a_1$$

and for $n = 2k$

$$a_{2k} = \frac{2k}{2k-1} \cdot \frac{2k-2}{2k-3} \cdot \dots \cdot \frac{2}{1} \cdot a_0 = 2k \cdot \frac{2k-2}{2k-1} \cdot \frac{2k-4}{2k-3} \cdot \dots \cdot \frac{2}{3} \cdot \frac{1}{1} \cdot a_0$$

This can be written using double factorials

as

$$a_{2k-1} = \frac{(2k-1)!!}{(2k-2)!!} \cdot a_1$$

and

$$a_{2k} = 2k \cdot \frac{(2k-2)!!}{(2k-1)!!} \cdot a_0$$



Figure 1.1: QR-code
wikipedia

So

$$(2k-1) \cdot \frac{((2k-2)!!)^2}{((2k-1)!!)^2} \cdot a_0 \leq a_1 \leq 2k \cdot \frac{((2k-2)!!)^2}{((2k-1)!!)^2} \cdot a_0$$

As we can easily see

$$a_1 = \lim_{k \rightarrow \infty} 2k \cdot \frac{((2k-2)!!)^2}{((2k-1)!!)^2} \cdot a_0$$

From the properties of double factorials it follows that

$$(2k-2)!! = 2^{k-1} \cdot (k-1)! = 2^{k-1} \cdot \Gamma(k)$$

and

$$(2k-1)!! = \frac{2^k}{\sqrt{\pi}} \Gamma\left(\frac{1}{2} + k\right)$$

So with a well-known limit we get

$$a_1 = \lim_{k \rightarrow \infty} \pi \cdot \frac{k(\Gamma(k))^2}{(\Gamma(\frac{1}{2} + k))^2} = \pi \cdot 1 = \pi$$

Solution 2

Without using double factorials we can write

$$a_{2k-1} = \frac{(2k-1)!}{(2^{k-1} \cdot (k-1)!)^2} \cdot a_1$$

and

$$a_{2k} = \frac{2k(2^{k-1}(k-1)!)^2}{(2k-1)!} \cdot a_0$$

and hence

$$a_1 = \lim_{k \rightarrow \infty} \frac{2k(2^{k-1}(k-1)!)^4}{((2k-1)!)^2} \cdot a_0 = \lim_{k \rightarrow \infty} \frac{2k \cdot 2^{4(k-1)}}{(2k-1)^2 \binom{2k-2}{k-1}^2} \cdot a_0$$

Writing $a_0 = 2$ and $n = k - 1$ we get with other well-known limits

$$a_1 = \lim_{n \rightarrow \infty} \frac{4(n+1) \cdot 2^{4n}}{(2n+1)^2 \binom{2n}{n}^2} = \lim_{n \rightarrow \infty} \frac{4n^2 + 4n}{4n^2 + 4n + 1} \cdot \frac{2^{4n}}{n \binom{2n}{n}^2} = 1 \cdot \pi = \pi$$

2

Opgave B NAW 5/4 nr. 3, September 2003

The problem

Introduction.

Let $S(n)$ be the sum of the remainders on division of the natural number n by $2, 3, \dots, n-1$. Show that

$$\lim_{n \rightarrow \infty} \frac{S(n)}{n^2}$$

exists and compute its value.

Solution.

We define the remainder in the division of n by k by $r_{n,k} = n - k \cdot \lfloor \frac{n}{k} \rfloor$, so from the definition of $S(n)$ it follows that

$$S(n) = \sum_{k=2}^{n-1} r_{n,k} = \sum_{k=1}^n r_{n,k} = \sum_{k=1}^n (n - k \cdot \lfloor \frac{n}{k} \rfloor) = n^2 - \sum_{k=1}^n k \cdot \lfloor \frac{n}{k} \rfloor$$

With $\sigma(k) = \sum_{d|k} d$, Theorem 324 and the proof of this Theorem taken from Hardy and Wright, *An Introduction to the Theory of Numbers*, 5th ed. p. 264-266¹, we get

$$\begin{aligned} S(n) &= n^2 - \sum_{x=1}^n \sum_{1 \leq y \leq n/x} y = n^2 - \sum_{k=1}^n \sigma(k) \\ &= n^2 - \left(\frac{1}{12} \pi^2 n^2 + O(n \log n) \right) \end{aligned}$$

Hence

$$\lim_{n \rightarrow \infty} \frac{S(n)}{n^2} = 1 - \frac{1}{12} \pi^2$$

¹ G.H. Hardy and E.M. Wright. *An Introduction to the Theory of Numbers*. Clarendon Press Oxford University Press, Oxford New York, 1979. ISBN 0198531710

3

*Opgave C NAW 5/4 nr. 3, September 2003**The problem**Introduction*

See NAW 5/4 nr. 3 September 2003, problem C:

<https://www.nieuwarchief.nl/serie5/pdf/naw5-2003-04-3-269.pdf>

Solution

Let the points A, B, C , and D be defined by coordinates (x_1, y_1) , (x_2, y_2) , (x_3, y_3) , and (x_4, y_4) . As can easily be shown, we have $Z_1 = Z_2$ with $x_{Z_1} = x_{Z_2} = \frac{1}{4} \sum_{i=1}^4 x_i$ and $y_{Z_1} = y_{Z_2} = \frac{1}{4} \sum_{i=1}^4 y_i$.

Let S_1 be the centre of gravity of ABD and S_2 that of BCD , then we have

$$x_{S_1} = \frac{x_1 + x_2 + x_4}{3}, y_{S_1} = \frac{y_1 + y_2 + y_4}{3}, x_{S_2} = \frac{x_2 + x_3 + x_4}{3}, \text{ and } y_{S_2} = \frac{y_2 + y_3 + y_4}{3}.$$

We define A_1 as the area of ABD , A_2 the area of BCD , and the weighting factors $p_1 = \frac{A_1}{A_1 + A_2}$ and $p_2 = \frac{A_2}{A_1 + A_2}$.

According to a more or less well-known result we can calculate A_1 and A_2 from the coordinates:

$$A_1 = \frac{1}{2}((y_2 - y_1)(x_1 + x_2) + (y_4 - y_2)(x_4 + x_2) + (y_1 - y_4)(x_1 + x_4)) \text{ and}$$

$$A_2 = \frac{1}{2}((y_3 - y_2)(x_3 + x_2) + (y_4 - y_3)(x_4 + x_3) + (y_2 - y_4)(x_2 + x_4)).$$



Figure 3.1: QR-code link

In this article you find my published solution of Opgave 2003/1A. This could have been Chapter 1.

See Lemma below

Now we can calculate Z_3 with $x_{Z_3} = p_1 \cdot x_{S_1} + p_2 \cdot x_{S_2}$ and $y_{Z_3} = p_1 \cdot y_{S_1} + p_2 \cdot y_{S_2}$.

If $Z_1 = Z_2 = Z_3$ we have equations

$$(4p_1 - 3)x_1 + x_2 + (1 - 4p_1)x_3 + x_4 = 0 \quad (3.1)$$

and

$$(4p_1 - 3)y_1 + y_2 + (1 - 4p_1)y_3 + y_4 = 0 \quad (3.2)$$

Without loss of generality, we may assume that $A(-a, 0)$, $B(0, b)$, $C(x_3, y_3)$, and $D(0, d)$, with $a > 0$ and $d > b$. We have $A_1 = \frac{1}{2}a(d - b)$, $A_2 = \frac{1}{2}x_3(d - b)$ and $p_1 = \frac{a}{a+x_3}$.

Solving equation (3.1) for x_3 gives $x_3 = \pm a$. The only valid solution is $x_3 = a$. With the second equation we find $y_3 = b + d$.

So ABCD is a parallelogram.

Lemma

The area A of a simple region R can be calculated by an integral over the boundary C of R :

$$A = \oint_C (x \, dx + y \, dy)$$

Proof: We use Green's Theorem:

$$\oint_C (P \, dx + Q \, dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy$$

Suppose $Q(x, y) = P(x, y) = x$, then we get:

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

So the right-hand side equals the area A of the region R . Let $P_0, P_1, P_2, \dots, P_n$, where $P_n = P_0$, be the polygon vertices. The area of the polygon can be calculated by the circular integral over the sides:

$$A = \oint_C (x \, dx + y \, dy) = \sum_{i=0}^{n-1} I_i$$

Here, I_i is the contribution to the integral along the line segment $P_i P_{i+1}$. P_i has coordinates (x_i, y_i) .

Suppose $x_{i+1} \neq x_i$, then the equation of the line through P_i and P_{i+1} is

$$y - y_i = \frac{y_{i+1} - y_i}{x_{i+1} - x_i} (x - x_i)$$

so

$$dy = \frac{y_{i+1} - y_i}{x_{i+1} - x_i} dx$$

Now we calculate the integral I_i :

$$\begin{aligned} I_i &= \int_{x_i}^{x_{i+1}} (x dx + x dy) \\ &= \int_{x_i}^{x_{i+1}} x dx + \int_{x_i}^{x_{i+1}} x \frac{y_{i+1} - y_i}{x_{i+1} - x_i} dx \\ &= \left(1 + \frac{y_{i+1} - y_i}{x_{i+1} - x_i}\right) \int_{x_i}^{x_{i+1}} x dx \\ &= \left(1 + \frac{y_{i+1} - y_i}{x_{i+1} - x_i}\right) \cdot \frac{1}{2} (x_{i+1}^2 - x_i^2) \\ &= \frac{1}{2} (x_{i+1}^2 - x_i^2) + \frac{1}{2} (y_{i+1} - y_i) (x_{i+1} + x_i) \end{aligned}$$

The same formula applies when $x_{i+1} = x_i$.

Now we calculate $A = \sum_{i=0}^{n-1} I_i$. Since $x_n = x_0$, quadratic terms cancel out.

Conclusion:

$$A = \frac{1}{2} \sum_{i=0}^{n-1} (y_{i+1} - y_i) (x_{i+1} + x_i) \quad (3.3)$$

Example

The triangle with vertices $(0,0)$, $(4,3)$, and $(3,5)$ has an area

$$A = \frac{1}{2} ((3-0)(0+4) + (5-3)(4+3) + (0-5)(3+0)) = \frac{12+14-15}{2} = \frac{11}{2}.$$

Which can be verified by elementary means.

4

*Opgave A NAW 5/4 nr. 4, December 2003**The problem**Introduction*

For each non-negative integer n , let a_n be the number of digits in the decimal expansion of 2^n that are at least 5. Evaluate the sum $\sum_{n=0}^{\infty} \frac{a_n}{2^n}$.



Figure 4.1: NAW December 2003

Solution

Let $b(n)$ be the number of odd digits in the decimal expansion of 2^n . We can easily see that (change of notation) $a(n) = b(n+1)$, because a digit with value 5 or higher in 2^n generates an odd digit in the next generation 2^{n+1} . The sequence $b(n)$ is well known from Sloane's On-Line Encyclopedia of Integer Sequences as A055254. See [1] and [2]. We evaluate

$$\sum_{n=0}^{\infty} \frac{a(n)}{2^n} = \sum_{n=0}^{\infty} \frac{b(n+1)}{2^n}$$

We do not know a formula for $a(n)$ nor $b(n)$ other than an algorithm that can be implemented for instance in Maple.

```
A055254:=proc(n) local i, j, k, val;
val:= 2^n; j:=0; k:= floor(ln(val)/ln(10))+1;
for i from 1 to k do
  if (val mod 10) mod 2 = 1 then j:=j+1 fi;
```


5

Opgave B NAW 5/4 nr. 4, December 2003

The problem

Introduction.

Let G be a group such that squares commute and cubes commute,

i.e., $g^2h^2 = h^2g^2$ and $g^3h^3 = h^3g^3$ for all $g, h \in G$.

Show that G is Abelian.

Solution.

We define $(x, y) = x^{-1}y^{-1}xy$, called commutator of x and y . From this definition follows $(x, y) = 1$ if, and only if, $xy = yx$. Thus all commutators in group G are 1 if, and only if, G is an Abelian group. The subgroup G' of G generated by all commutators (x, y) is called the commutator subgroup or derived group. The factor group G/G' is Abelian.

Our problem can be translated in the statement: The factor group G/K is Abelian if K is the group generated by the commutators (x^2, y^2) and (x^3, y^3) with x and y in G . Or $(x^2, y^2) = 1 \wedge (x^3, y^3) = 1$ implies $(x, y) = 1$.

Let G be the free group generated by ' a ' and ' b '. Can we prove that the factor group $G/[(a^2, b^2), (a^3, b^3)]$ is Abelian?

See <https://www.nieuwarchief.nl/serie5/pdf/naw5-2004-05-2-174.pdf> for a real proof.



Figure 5.1: NAW December 2003



Figure 5.2: NAW June 2004

Remark.

This kind of problem is not very interesting. The solution just shows you how to manipulate some expressions! It is nothing more than a trick. But the result is useful! See Chapter 17.

6

Opgave C NAW 5/4 nr. 4, December 2003

The problem

Introduction.

Let $(X_n)_{n \geq 1}$ be a sequence of independent and identically distributed random variables with $P\{X_n = 1\} = P\{X_n = -1\} = \frac{1}{2}$. Set $S_n = \sum_{k=1}^n X_k$. Calculate $P\{S_3 = 1 \vee S_6 = 2 \vee \dots \vee S_{3n} = n \vee \dots\}$.

Solution.

Let $\mathcal{P}(n) = P\{S_3 = 1 \vee S_6 = 2 \vee \dots \vee S_{3n} = n\}$ and $A_n = \{1, 2, 3, \dots, n\}$. We notice that

$$P\{S_{3k} = k\} = \frac{\binom{3k}{k}}{2^{3k}}$$

With the principle of inclusion/exclusion we get

$$\mathcal{P}(n) = P(n, 1) - P(n, 2) + \dots + (-1)^{k-1} P(n, k) + \dots + (-1)^{n-1} P(n, n)$$

where

$$\begin{aligned} P(n, k) &= \sum_{\{i_1, i_2, \dots, i_k\} \subset A_n} P\{S_{3i_1} = i_1 \vee S_{3i_2} = i_2 \vee \dots \vee S_{3i_k} = i_k\} = \\ &= \sum_{i_1 < i_2 < \dots < i_k \leq n} \frac{\binom{3i_1}{i_1} \binom{3i_2 - 3i_1}{i_2 - i_1} \dots \binom{3i_k - 3i_{k-1}}{i_k - i_{k-1}}}{2^{3i_1} 2^{3i_2 - 3i_1} \dots 2^{3i_k - 3i_{k-1}}} = \\ &= \sum_{i_1 < i_2 < \dots < i_k \leq n} \frac{\binom{3i_1}{i_1} \binom{3i_2 - 3i_1}{i_2 - i_1} \dots \binom{3i_k - 3i_{k-1}}{i_k - i_{k-1}}}{2^{3i_k}} \end{aligned}$$

We have to calculate $\lim_{n \rightarrow \infty} \mathcal{P}(n)$.

We see that

$$\begin{aligned} P(n+1, k) &= \sum_{i_1 < i_2 < \dots < i_k \leq n+1} \frac{\binom{3i_1}{i_1} \binom{3i_2-3i_1}{i_2-i_1} \dots \binom{3i_k-3i_{k-1}}{i_k-i_{k-1}}}{2^{3i_k}} = \\ &= P(n, k) + \sum_{i_1 < i_2 < \dots < i_{k-1} < i_k = n+1} \frac{\binom{3i_1}{i_1} \binom{3i_2-3i_1}{i_2-i_1} \dots \binom{3n+3-3i_{k-1}}{n+1-i_{k-1}}}{2^{3n+3}} \end{aligned}$$

and so

$$\mathcal{P}(n+1) = \mathcal{P}(n) + \mathcal{D}(n)$$

with

$$\mathcal{D}(n) = \sum_{k=1}^{n+1} (-1)^{k-1} \sum_{i_1 < i_2 < \dots < i_{k-1} < i_k = n+1} \frac{\binom{3i_1}{i_1} \binom{3i_2-3i_1}{i_2-i_1} \dots \binom{3n+3-3i_{k-1}}{n+1-i_{k-1}}}{2^{3n+3}}$$

We have $\mathcal{P}(2) = \mathcal{P}(1) + \mathcal{D}(1)$ and $\mathcal{P}(3) = \mathcal{P}(2) + \mathcal{D}(2) = \mathcal{P}(1) + \mathcal{D}(1) + \mathcal{D}(2)$, etcetera. Hence

$$\mathcal{P}(n) = \mathcal{P}(1) + \sum_{i=1}^{n-1} \mathcal{D}(i) = \frac{3}{8} + \sum_{i=1}^{n-1} \mathcal{D}(i)$$

Elementary counting gives the following results:

$$\begin{aligned} \mathcal{D}(1) &= 6/64 = 0.093750 \\ \mathcal{D}(2) &= 21/512 = 0.041016 \\ \mathcal{D}(3) &= 90/4096 = 0.021973 \\ \mathcal{D}(4) &= 429/32768 = 0.013092 \\ \mathcal{D}(5) &= 2184/262144 = 0.008331 \\ \mathcal{D}(6) &= 11628/2097152 = 0.005545 \\ \mathcal{D}(7) &= 63954/16777216 = 0.003812 \\ \mathcal{D}(8) &= 360525/134217728 = 0.002686 \\ \mathcal{D}(9) &= 2072070/1073741824 = 0.001930 \\ \mathcal{D}(10) &= 12096045/8589934592 = 0.001408 \end{aligned}$$

Total $\sum_{i=1}^{10} \mathcal{D}(i) = 1662515613/8589934592$, so

$$\mathcal{P}(11) = \frac{3}{8} + \sum_{i=1}^{10} \mathcal{D}(i) = 4883741085/8589934592 = 0.5685422901$$

We can do better: the sequence $a(n)_{n \geq 1} = 6, 21, 90, 429, 2184, 11628, \dots$ can be written as:

$$a(n) = \frac{2}{3n+2} \binom{3n+3}{n+1}$$

and hence

$$\mathcal{D}(n) = \frac{a(n)}{2^{3n+3}}$$

Further we can write

$$\mathcal{P}(n) = \frac{3}{8} + \sum_{i=1}^{n-1} \mathcal{D}(i) = \frac{3}{8} + \sum_{i=1}^{n-1} \frac{\binom{3i+3}{i+1}}{(3i+2)2^{3i+2}} \quad (6.1)$$

The probability in question is

$$\lim_{n \rightarrow \infty} \mathcal{P}(n) = \frac{3}{8} + \sum_{i=1}^{\infty} \frac{\binom{3i+3}{i+1}}{(3i+2)2^{3i+2}} = 0.57294901687515772769311 \dots$$

Conclusion.

The above calculations are based on a lemma:

$$\mathcal{D}(n) = \frac{2}{3n+2} P\{S_{3n+3} = n+1\} \quad (6.2)$$

This lemma can be proved with induction on n , proving

$$a(n) = 2^{3n+3} \cdot \mathcal{D}(n) = \frac{2}{3n+2} \binom{3n+3}{n+1} = \frac{3(3n+1)}{(2n+1)(n+1)} \binom{3n}{n}$$

The summand

$$\mathcal{D}(i) = \frac{\binom{3i+3}{i+1}}{(3i+2)2^{3i+2}}$$

is a hypergeometric term, but not 'Gosperable', so there is no closed form for $\mathcal{P}(n)$ in the sense of [1]¹ Definition 8.1.1. See [1] and [2]. Maple 8 gives a ${}_3F_2$ hypergeometric form.

$$\lim_{n \rightarrow \infty} \mathcal{P}(n) = \frac{3}{8} + \sum_{i=1}^{\infty} \frac{\binom{3i+3}{i+1}}{(3i+2)2^{3i+2}} = \frac{3}{8} + \frac{3}{32} \cdot {}_3F_2\left(1, \frac{5}{3}, \frac{7}{3}; \frac{5}{2}, 3; \frac{27}{32}\right)$$

which evaluates to $0.57294901687515772769311 \dots$.

¹ Marko Petkovšek, Herbert Wilf, and Doron Zeilberger. *A = B*. A K Peters/CRC Press, Natick, MA, USA, 1996. ISBN 1568810636

References

- [1] Petkovšek, Wilf and Zeilberger, $A = B$, A.K. Peters, Massachusetts, 1996.
- [2] The Maple SumTools[Hypergeometric] library

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Opgave A NAW 5/5 nr. 1, March 2004

The problem

Introduction

For every integer $n > 2$ prove that

$$\sum_{j=1}^{n-1} \left(\frac{1}{n-j} \sum_{k=j}^{n-1} \frac{1}{k} \right) < \frac{\pi^2}{6}$$

Solution

Let

$$s_{n-1} = \sum_{j=1}^{n-1} \left(\frac{1}{n-j} \sum_{k=j}^{n-1} \frac{1}{k} \right) \quad (7.1)$$

We have $s_1 = 1$, $s_2 = 1\frac{1}{4}$, $s_3 = \frac{49}{36} = \frac{5}{4} + \frac{1}{9}$, and $s_4 = s_3 + \frac{1}{42}$.

Proposition

$$s_n = s_{n-1} + \frac{1}{n^2} \quad \text{for } n > 1 \quad (7.2)$$

From this proposition follows:

$$s_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} = \sum_{k=1}^n \frac{1}{k^2} \quad (7.3)$$

And we are finished, because $\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{k^2} = \frac{\pi^2}{6}$, we have $s_{n-1} < s_n < \frac{\pi^2}{6}$. Note: this is true for $n \geq 2$.

Now we prove the proposition.

Let $A_{n-1} = (a_{ij}) = \left(\frac{1}{n-i} \cdot \frac{1}{j}\right)$ with $1 \leq i \leq j \leq n-1$ and

$B_n = (b_{ij}) = \left(\frac{1}{n+1-i} \cdot \frac{1}{j}\right)$ with $1 \leq i \leq j \leq n$.

Then

$$s_{n-1} = \sum_{1 \leq i \leq j \leq n-1} a_{ij} \quad \text{and} \quad s_n = \sum_{1 \leq i \leq j \leq n} b_{ij}$$

Comparing a_{ij} with b_{ij} we see $b_{ij} = a_{i-1,j}$ for $2 \leq i \leq j \leq n-1$.

So

$$s_n = s_{n-1} - \sum_{i=1}^{n-1} a_{ii} + \sum_{j=1}^n b_{1j} + \sum_{i=1}^n b_{in} - b_{nn}$$

We can write $a_{ii} = \frac{1}{(n-i)i} = \frac{1}{n(n-i)} + \frac{1}{ni}$, so

$$\sum_{i=1}^{n-1} a_{ii} = \frac{1}{n} \sum_{i=1}^{n-1} \frac{1}{n-i} + \frac{1}{n} \sum_{i=1}^{n-1} \frac{1}{i} = \frac{2}{n} H_{n-1}$$

where H_{n-1} is the $(n-1)$ -th harmonic number.

Further we know

$$\sum_{j=1}^n b_{1j} = \sum_{i=1}^n b_{in} = \frac{1}{n} \sum_{k=1}^n \frac{1}{k} = \frac{1}{n} H_n$$

Now

$$s_n = s_{n-1} - \frac{2}{n} H_{n-1} + \frac{1}{n} H_n + \frac{1}{n} H_n - \frac{1}{n^2}$$

and hence

$$s_n = s_{n-1} + \frac{2}{n} (H_n - H_{n-1}) - \frac{1}{n^2} = s_{n-1} + \frac{2}{n} \cdot \frac{1}{n} - \frac{1}{n^2}$$

This concludes the proof of the proposition.

$$s_n = s_{n-1} + \frac{1}{n^2} \tag{7.4}$$

See for instance Graham, Knuth, Patashnik, *Concrete Mathematics*, p. 258, section 6.3.

8

*Opgave B NAW 5/5 nr. 1, March 2004**The problem**Introduction*

Consider the first digit in the decimal expansion of 2^n for $n \geq 0$:
 1, 2, 4, 8, 1, 3, 6, 1, 2, 5, 1, 2, 4, \dots . Does the digit 7 appear in this
 sequence? Which digit appears more often, 7 or 8? How many
 times more often?

Solution

The first question is easily answered in the affirmative: $2^{46} =$
 70368744177664.

The sequence is well known from Sloane's On-Line Encyclopedia
 of Integer Sequences as A008952. See [1]. We use the formula

<https://oeis.org/A008952>

$$a(n) = \lfloor 2^n / 10^{\lfloor n \cdot \frac{\ln 2}{\ln 10} \rfloor} \rfloor$$

in a Maple program [2] to calculate the frequency of digit d in all
 $a(k)$ with $k \leq n$.

```
> A:=proc(n,d) local i,k,s;
  s:=0;
  for k from 0 to n do
    i:=floor(2^k / 10^floor(k*ln(2)/ln(10)));
    if i=d then s:=s+1 fi;
  od;
  RETURN(s);
end;
```

The second question can be answered as: *'the winner is 7'*! From $n > 209$ or slightly more, the frequency of the digit 7 exceeds that of 8.

We can generalize: for large enough n , the frequencies form a decreasing sequence. That is, for digits $d_1 < d_2$, we have $f(d_1) > f(d_2)$.

But Benford's law comes to the rescue (see [3]), our sequence is a well-known example:

$$\text{Prob}(\text{first significant digit} = d) = \log_{10}\left(1 + \frac{1}{d}\right), \text{ for } d = 1, 2, \dots, 9$$

The last question is the most difficult to answer. Our best guess for large n is according to Benford's law:

$$\frac{f(7)}{f(8)} \text{ tends to } \frac{\log_{10}(1 + 1/7)}{\log_{10}(1 + 1/8)} = 1.133706496$$

References

[1] <https://oeis.org>

[2] <https://www.maplesoft.com/>

[3] Hill, T.P., The Significant-Digit Phenomenon, Amer. Math. Monthly 102, 322-327, 1995

9

*Opgave A NAW 5/5 nr. 2, June 2004**The problem**Introduction*

The sequence 333111333131333111333... is identical to the sequence of its block lengths. Compute the frequency of the number 3 in this sequence.

Solution

This sequence is known as the Kolakoski-(3,1) sequence. See Neil Sloane's On-Line Encyclopedia of Integer Sequences, sequence number A064353, which is in fact the Kolakoski-(1,3) sequence, differing only in the first position. See [1].

<https://oeis.org/A064353>

Michael Baake and Bernd Sing wrote: Unlike the (classical) Kolakoski sequence on the alphabet {1,2}, its analogue on {1,3} can be related to a primitive substitution rule. See [2] and [3]. We base our calculations on section 2 of their paper.

Let $A = 33$, $B = 31$ and $C = 11$. In the case of Kol(3,1) the substitution σ and the matrix M of the substitution are given by

$$\begin{array}{l} A \mapsto ABC \\ \sigma: B \mapsto AB \\ C \mapsto B \end{array} \quad \text{and} \quad M = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad (9.1)$$

where $m_{ij} = 1$ if and only if there is a corresponding mapping in σ , for instance $A \mapsto ABC$ corresponds to the first column of M , etc.

An infinite fixed point can be obtained as follows:

$$A \mapsto ABC \mapsto ABCABB \mapsto \dots \quad (9.2)$$

This corresponds to

$$333111333131\dots \quad (9.3)$$

which is the unique infinite Kol(3,1) sequence. The matrix M is primitive because M^3 has only positive entries. The characteristic polynomial $P(\lambda)$ of M is

$$P(\lambda) = \lambda^3 - 2\lambda^2 - 1, \quad (9.4)$$

and has one real root λ_1 and two complex roots $\lambda_{2,3}$. We have

$$2.205569 \approx \lambda_1 > 1 > |\lambda_2| = |\lambda_3| \approx 0.67 \quad (9.5)$$

According to the Perron-Frobenius Theorem there is a positive eigenvector corresponding to λ_1 . We easily verify that $\mathbf{x}_1 = (\lambda_1, \lambda_1^2 - \lambda_1, 1)^T$ is such an eigenvector.

Starting with $\mathbf{x}(0) = (1, 0, 0)^T$ we define

$$\mathbf{x}(k+1) = M\mathbf{x}(k) \quad (9.6)$$

The asymptotic behavior of this system will be of the form $\mathbf{x}(n) = c \cdot (\lambda_1)^n \mathbf{x}_1$ for some value of c .

From $\mathbf{x}(n)$ we can calculate the number of A 's, B 's and C 's. In $A = 33$ there are two 3 's, etc., so we can easily calculate the relative frequencies of the letters of the alphabet. The frequency of the digit '3' is:

$$\rho_3 = \frac{2 \cdot \lambda_1 + 1 \cdot (\lambda_1^2 - \lambda_1) + 0 \cdot 1}{2 \cdot (\lambda_1^2 + 1)} \approx 0.6027847150 \quad (9.7)$$

References

- [1] <https://oeis.org/A064353>
- [2] Baake, Sing: Kolakoski-(3,1) is a (deformed) Model Set, *Canad. Math. Bull.* 47, No. 2, 168–190 (2004)
- [3] See also <https://arxiv.org/abs/math.MG/0206098>

See Wikipedia



Figure 9.1: Wikipedia Page

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Opgave C NAW 5/5 nr. 2, June 2004

The problem

Introduction

Let A be a ring and let $B \subset A$ be a subring. As a subgroup, B has finite index in A . Show that there exists a two-sided ideal I of A such that $I \subset B$ and I has finite index as a subgroup of A .

Solution

We have A and B as defined above. The index $[A : B] = k < \infty$ or in other words, the additive factor group A/B is a finite abelian group built from cosets of the form $x + B$.

Let $\mathcal{E}(G)$ be the ring of endomorphisms of the abelian group G . We define a ring homomorphism $f: B \rightarrow \mathcal{E}(A/B)$: for $a \in B$ we define $f: a \mapsto \alpha$ with $(x + B)\alpha = xa + B$. Note that we use the *right* function notation here, avoiding the notion of anti-homomorphism (see [2]).

The kernel of f is $L = \{a \in B \mid Aa \subset B\}$, L is the largest left ideal of A with $L \subset B$. The factor group B/L is isomorphic to a subgroup of $\mathcal{E}(A/B)$, so B/L is a finite abelian group, and since $(A/L)/(B/L) \cong A/B$ it follows that A/L is finite abelian.

We now consider the ring homomorphism $g: A \rightarrow \mathcal{E}(A/L)$: for $b \in A$ we define $g: b \mapsto \beta$ with $\beta(x + L) = bx + L$. Its restriction to L , $g_L: L \rightarrow \mathcal{E}(A/L)$ has kernel

$$I = \{a \in L \mid aA \subset L\} = \{a \in B \mid Aa \subset B \wedge aA \subset B\}.$$

I is the largest two-sided ideal of A with $I \subset B$. We have L/I finite and hence A/I is a finite abelian group, so $[A : I] < \infty$.

References

[1] ¹ Marshall Hall, Jr. *The Theory of Groups*, Macmillan, New York, 1959.

[2] <https://planetmath.org/encyclopedia/UnitalModule.html>

Look up this URL in the Wayback Machine. It may or may not help you!

¹ Jr Hall, Marshall. *The Theory of Groups*. The Macmillan Company, New York, 1959

11

*Problem A NAW 5/5 nr. 4, December 2004**The problem**Introduction*

1. Show that there exist infinitely many $n \in \mathbf{N}$, such that $S_n = 1 + 2 + \dots + n$ is a square.
2. Let a_1, a_2, a_3, \dots be those squares. Calculate $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$.

Solution

We know $S_n = \frac{1}{2}n(n+1)$ so we have to solve the diophantine equation

$$\frac{1}{2}n(n+1) = m^2 \quad (11.1)$$

Rewriting gives $4n^2 + 4n = 8m^2$ or $(2n+1)^2 - 1 = 2(2m)^2$.
Substituting $x = 2n+1$ and $y = 2m$ we get the Pell equation

$$x^2 - 2y^2 = 1 \quad (11.2)$$

with an infinite number of solutions $(3, 2), (17, 12), (99, 70) \dots$ with corresponding $n = 1, 8, 49, 288, \dots$

A well-known result gives solutions of (2)

$$x_k = \frac{(3 + 2\sqrt{2})^k + (3 - 2\sqrt{2})^k}{2}$$

and

$$y_k = \frac{(3 + 2\sqrt{2})^k - (3 - 2\sqrt{2})^k}{2\sqrt{2}}$$

The sequence $\{a_i\}_{i=1,2,3,\dots}$ starts with 1, 36, 1225, 41616, ... and can be calculated with

$$a_k = \frac{y_k^2}{4} = \frac{((3 + 2\sqrt{2})^k - (3 - 2\sqrt{2})^k)^2}{32} \quad (11.3)$$

which can be rewritten as

$$a_k = \frac{((17 + 12\sqrt{2})^k + (17 - 12\sqrt{2})^k) - 2}{32} \quad (11.4)$$

And so

$$\frac{a_{k+1}}{a_k} = \frac{((17 + 12\sqrt{2})^{k+1} + (17 - 12\sqrt{2})^{k+1}) - 2}{((17 + 12\sqrt{2})^k + (17 - 12\sqrt{2})^k) - 2} \quad (11.5)$$

We easily see that $\lim_{k \rightarrow \infty} \frac{a_{k+1}}{a_k} = 17 + 12\sqrt{2}$.

Remark

Finding a triangular number S_n that is cubic, except the trivial 1, would be spectacular. As we try to solve

$$\frac{1}{2}n(n+1) = m^3$$

substituting $X = 2m$ and $Y = 2n + 1$ we get the elliptic curve with equation

$$Y^2 = X^3 + 1 \quad (11.6)$$

We find this curve as A36 in the Cremona table. The torsion group is of order 6 with real members $(-1,0)$, $(0,-1)$, $(0,1)$, $(2,-3)$ and $(2,3)$. This means the only cubic triangular number is 1.

Moreover the above equation is also known from the Catalan conjecture, or should we say Catalan theorem: the only non-trivial integer powers that differ 1 are 2^3 and 3^2 .

References

[1] <https://oeis.org/A001110>

12

*Problem B NAW 5/5 nr. 4, December 2004**The problem**Introduction.*

Let G be a finite set of elements and \cdot a binary associative operation on G . There is a neutral element in G and that is the only element in G with the property $a \cdot a = a$.

Show that G with the operation \cdot is a group.

Solution.

G is a finite semigroup with identity. Let A be a subset of G . There is a smallest subsemigroup K of G which contains A . We say A generates K , notation $\langle A \rangle = K$. A single element x of G generates a subsemigroup $\langle x \rangle = \{x^n \mid n > 0\}$. Since $\langle x \rangle$ is finite there must be integers $p > q$, such that $x^p = x^q$. So $x^p = x^{q+k} = x^q x^k = x^k x^q = x^q$ and $e = x^k$ is a neutral element for $\langle x \rangle$. We assume that k is the smallest integer with this property.

We easily verify that $\langle x \rangle = \{e, x, x^2, \dots, x^{k-1}\}$ is a group with neutral element e and as such a subgroup of G . Clearly e is idempotent with $e \cdot e = e^2 = e$. According to the problem statement e is the only element of G with this property.

We now prove the following lemma:

Let G be a finitely generated semigroup and H a subgroup of G . Then there exists a maximal subgroup M of G containing H .

Proof: Let G be generated by x_1, \dots, x_m and let y_1 be the first of the x_i not contained in H and with property $H_1 = \{H, y_1\}$ is a group. If such a y_1 does not exist then $M = H$ is the maximal subgroup of G . We now have $H_1 \supseteq H$. If $H_1 = G$, then G is the maximal subgroup sought. If not, choose $H_2 = \{H_1, y_2\} \supseteq H_1$, where y_2 is the first of the x_i not contained in H_1 and $\{H_1, y_2\}$ is a group. If such a y_2 does not exist then $M = H_1$ is the maximal subgroup of G .

Continuing this process we must reach the situation where no more extension is possible: $H_i \supseteq H_{i-1} \supseteq \dots \supseteq H$, H_i is a group. If $H_i = \{H_{i-1}, y_i\} = G$ the maximal subgroup is G else the maximal subgroup $M = H_i$ is a proper subgroup of G .

G is finite and so certainly finitely generated. According to the above lemma $\{x\}$ is contained in a maximal subgroup M . If $M = G$ we are ready, but let there be a y not in M , then $\{y\}$ is contained in a maximal subgroup M' , with neutral element e' , with $e' \cdot e' = e'$. If $e' \neq e$ we have a contradiction and there is no such element y , hence $M = G$. If $e' = e$ then we easily see that $\{M, y\}$ is a group in contradiction with the maximality of M . So we have proved that G is a group.

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Problem A NAW 5/6 nr. 1, March 2005

The problem

Introduction

Calculate

$$\sum_{n=1}^{\infty} \frac{1}{\sum_{i=1}^n i^2}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{\sum_{i=1}^n i^3}$$

Solution

These kinds of problems make me feel young. They remind me of the early sixties and the lectures of Prof. Van der Blij.

Part 1

By a well-known result we first write $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$ and hence the first summand can be written as

$$\frac{6}{n(n+1)(2n+1)} = \frac{6}{n} + \frac{6}{n+1} - \frac{24}{2n+1}.$$

Let

$S_n = \sum_{k=1}^n \frac{1}{\sum_{i=1}^k i^2} = 6 \sum_{k=1}^n \frac{1}{k} + 6 \sum_{k=1}^n \frac{1}{k+1} - 24 \sum_{k=1}^n \frac{1}{2k+1}$, so using a result on harmonic numbers we get

$$S_n = 6H_n + 6(H_n - 1) - 24(H_{2n+1} - \frac{1}{2}H_n - 1) = 18 - 24(H_{2n+1} - H_n)$$



Figure 13.1: Wikipedia

H_n being the n -th harmonic number. We know $H_n = \ln n + \Delta_n$ with $\lim_{n \rightarrow \infty} \Delta_n = \gamma$, Euler's constant.

Now with $H_{2n+1} - H_n = \ln(2n+1) - \ln n - \Delta_{2n+1} + \Delta_n$ we can easily see that $\lim_{n \rightarrow \infty} (H_{2n+1} - H_n) = \ln 2$ and therefore the first answer is

$$\sum_{n=1}^{\infty} \frac{1}{\sum_{i=1}^n i^2} = 18 - 24 \ln 2$$

Part 2

First we write $\sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$ and hence the second summand can be written as

$$\frac{4}{n^2(n+1)^2} = \frac{4}{(k+1)^2} + \frac{4}{k^2} - \frac{8}{k} + \frac{8}{k+1}$$

Let

$$S_n = \sum_{k=1}^n \frac{1}{\sum_{i=1}^k i^3} = \sum_{k=1}^n \left(\frac{4}{(k+1)^2} + \frac{4}{k^2} - \frac{8}{k} + \frac{8}{k+1} \right), \text{ so}$$

$$S_n = 4 \sum_{k=1}^n \frac{1}{(k+1)^2} + 4 \sum_{k=1}^n \frac{1}{k^2} - 8 \sum_{k=1}^n \frac{1}{k} + 8 \sum_{k=1}^n \frac{1}{k+1}$$

and

$$S_n = 4 \left(\sum_{k=1}^n \frac{1}{k^2} - 1 \right) + 4 \sum_{k=1}^n \frac{1}{k^2} - 8H_n + 8(H_n - 1) = 8 \sum_{k=1}^n \frac{1}{k^2} - 12$$

So

$$\lim_{n \rightarrow \infty} S_n = 8 \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{k^2} - 12 = 8 \cdot \frac{1}{6} \pi^2 - 12 = \frac{4}{3} \pi^2 - 12$$

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*Problem C NAW 5/6 nr. 1, March 2005**The problem**Introduction*

We call a triangle integral if the sides of the triangle are integers. Consider the integral triangles with rational circumradius.

1. Prove that for any positive integer p there are only finitely many integers q such that there exists an integral triangle with circumradius equal to $\frac{p}{q}$.
2. Prove that for any positive integer q there exist infinitely many integral triangles with circumradius equal to $\frac{p}{q}$ for some integer p with $\gcd(p, q) = 1$.

Solution

Let triangle ABC have integer sides a, b and c with area A and circumradius R . There is a relation between these quantities given by Heron's formulas

$$(4A)^2 = (a + b + c)(a + b - c)(a - b + c)(-a + b + c)$$

and

$$A = \frac{abc}{4R} \quad \text{or equivalently} \quad R = \frac{abc}{4A}$$

So there is a one-to-one correspondence between the set of all integral triangles with rational/integer area and the set of all integral triangles with rational circumradius. The set of all integral triangles with integer area is well studied as the Heronian triangles.

Part 1

The sides being integers implies the existence of a minimal circumradius R_{\min} . For a given p there exist only finitely many q with $\frac{p}{q} > R_{\min}$. So there are only finitely many q such that $R = \frac{p}{q}$ is the circumradius of an integral triangle.

Part 2

The case $q = 1$ is trivial, as it is a well-known fact that there are infinitely many integers p such that p is the hypotenuse of a Pythagorean triangle. Scaling by two gives an integral triangle with circumradius $R = p$.

For the case $q > 1$ we use a parametric representation of the Heronian triangles as found in [1]

$$\begin{aligned} a &= n(m^2 + k^2) \\ b &= m(n^2 + k^2) \\ c &= (m + n)(mn - k^2) \\ A &= kmn(m + n)(mn - k^2) \end{aligned}$$

For any integers m, n and k with $mn > k^2 > \frac{m^2n}{2m+n}$, $\gcd(m, n, k) = 1$ and $m \geq n \geq 1$, we have one member of each similarity class of the Heronian triangles.

Using this and the formula above we get

$$R = \frac{(m^2 + k^2)(n^2 + k^2)}{4k}$$

In our case we do not need the restriction to unique reduced Heronian triangles. For the problem at hand we only need the triangle inequalities $a + b > c$, $a + c > b$ and $b + c > a$, together with $mn - k^2 > 0$. As we can easily see this can be realized by $m > k$, $n > k$ and $k \geq 1$.

Let $k = q$ and $p = \frac{(m^2 + q^2)(n^2 + q^2)}{4}$. All we have to prove is the existence of infinitely many (m, n) such that 4 divides $(m^2 + q^2)(n^2 + q^2)$. If q is even then choose $n > q$ with $\gcd(n, q) = 2$ so $4 \mid (n^2 + q^2)$, let $m > q$ be a positive integer with $\gcd(m, q) = 1$. If

q is not even choose $n > q$ and $m > q$ both odd with $\gcd(n, q) = \gcd(m, q) = 1$, so $2 \mid (n^2 + q^2)$ and $2 \mid (m^2 + q^2)$.

The sums $m^2 + q^2$ and $n^2 + q^2$ or $(\frac{n}{2})^2 + (\frac{q}{2})^2$ are so called primitive sums of two squares, defined by $x^2 + y^2$ with $\gcd(x, y) = 1$. For a prime divisor of such a primitive sum $x^2 + y^2$ it is not possible to be a divisor of y . In all cases we easily verify that we have a Heronian triangle with circumradius $R = \frac{p}{q}$ with $\gcd(p, q) = 1$, so we have infinitely many of them.

Reference

[1] Buchholz, R. H., Perfect Pyramids, Bull. Austral. Math. Soc. 45, nr 3, 1992.

15

*Problem A NAW 5/6 nr. 2, June 2005**The problem**Introduction*

A student association organises a large-scale dinner for 128 students. The chairs are numbered 1 through 128. The students are also assigned a number between 1 and 128. As the students come into the room one by one, they must sit at their assigned seat. However, one of the students is so drunk that he can't find his seat and takes an arbitrary one. Any sober student who comes in and finds his seat taken also takes an arbitrary one. The drunken student is one of the first 64 students. What is the probability that the last student gets to sit in the chair assigned to him?

Solution

We solve this problem for n students with $n > 1$. Without loss of generality we may assume that the first student is drunk, see below. There are three possibilities: student 1 sits on seat 1 (we call this success, because student n will be seated on seat n), student 1 sits on seat n (failure) or student 1 sits on a remaining arbitrary seat k .

In the last case the next students with numbers less than k will be seated on their assigned seat. Student k will now act as a drunken student by taking an arbitrary free seat. So in a way student k becomes the new number '1' of a corresponding

problem with $n - k + 1$ students. The choices are: Student '1' sits on seat 1 (success), or on the last seat (failure) or again the choice of an arbitrary free seat. This process is repeated until we have eventually two students, the 'first' and the last and there are only two choices, one leads to success, the other to failure.

At all stages of the process, success and failure have the same probability, so the probability the last student will be seated on the last chair is $1/2$.

See <https://www.nieuwarchief.nl/serie5/pdf/naw5-2005-06-4-332.pdf>



Figure 15.1: QR-code link

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*Problem C NAW 5/6 nr. 2, June 2005**The problem**Introduction*

In what follows, P stands for the set consisting of all odd prime numbers; M is the set consisting of all natural 2-powers $1, 2, 4, 8, 16, 32, \dots$; T is the set consisting of all positive integers that can be written as a sum of at least three consecutive natural numbers.

This problem plays a role in a few sequences in the OEIS: <https://oeis.org/A111774>, <https://oeis.org/A111775>, <https://oeis.org/A111787>, <https://oeis.org/A109814> and <https://oeis.org/A174090>

1. Show that the set theoretic union of P , M and T coincides with the set consisting of all the natural numbers..
2. Show that the sets P , M and T are pairwise disjoint.
3. Given $b \in T$, determine $t(b)$ in terms of the prime decomposition of b , where by definition $t(b)$ stands for the minimum of all those numbers $t > 2$ for which b admits an expression as sum of t consecutive natural numbers.
4. Consider the cardinality $C(b)$ of the set of all odd positive divisors of some element b of T . Now think of expressing this b in all possible ways as a sum of at least three consecutive natural numbers. Suppose this can be done in $S(b)$ ways. Determine the numerical connection between the numbers $C(b)$ and $S(b)$.

Remark: In this problem we clearly follow the convention not to include zero in the natural numbers.

Solution

First let

$$a = p_0^{e_0} \cdot p_1^{e_1} \cdots p_m^{e_m} \quad (16.1)$$

be the 'prime decomposition' of a positive integer a with $p_0 = 2$, $e_0 \geq 0$ and p_1, \dots, p_m odd primes with $e_i > 0$ for $i = 1, \dots, m$. We want to write a as the sum of k consecutive natural numbers starting with n .

$$a = n + (n+1) + \cdots + (n+k-1) = k \cdot n + \frac{k(k-1)}{2} = k(2n+k-1)/2$$

So

$$2a = k \cdot (2n+k-1) \quad (16.2)$$

We define k to be the smallest factor, thus $k < \sqrt{2a}$. We observe that only one of the factors is odd.

Part 1 and 2

When a is a power of 2 we can only have $k = 1$. A power of two is clearly not an odd prime and vice versa. An odd prime can only be written as a sum of 2 consecutive natural numbers ($k = 2$). For all other positive integers we have at least one odd prime divisor p_i . Let $k = p_i \geq 3$ and $n = (2a/k - k + 1)/2$. It follows that a can be written as the sum of at least three consecutive positive integers starting with n . The rest is trivial.

Part 3

Let again $b = a \in T$ and p_1 the smallest odd prime divisor of b . From (16.2) it follows that if $e_0 = 0$, meaning b is odd, we have $t(b) = p_1$, else $t(b) = \min(2^{e_0+1}, p_1)$.

Part 4

Let again $b = a \in T$. We use the prime decomposition (16.1) to find the number of all odd divisors of b . We easily see that this number must be $(e_1 + 1) \cdot (e_2 + 1) \cdots (e_m + 1)$.

So $C(b) = (e_1 + 1) \cdot (e_2 + 1) \cdots (e_m + 1)$.

$S(b)$ is the number of ways b can be expressed as sum of at least three positive integers.

From (16.2) it follows that for each odd divisor of b we can find a k with $k < \sqrt{2a}$. We must exclude $k = 1$ and $k = 2$.

Only in the case of an odd b can we have $k = 2$, so $S(b) = C(b) - 2$ if b is odd and $S(b) = C(b) - 1$ if b is even.

See <https://www.nieuwarchief.nl/serie5/pdf/naw5-2005-06-4-332.pdf>



Figure 16.1: QR-code link

17

*Problem B NAW 5/5 nr. 3, October 2005**The problem**Introduction.*

1. Let G be a group and suppose that the maps $f, g : G \rightarrow G$ with $f(x) = x^3$ and $g(x) = x^5$ are both group homomorphisms. Show that G is Abelian.
2. In the previous exercise, by which pairs (m, n) can $(3, 5)$ be replaced if we still want to be able to prove that G is Abelian.

*Solution**Part 1*

Let $(ab)^5 = a^5b^5$ for all $a, b \in G$, then we see that $(ba)^4 = a^4b^4$. Now $(ab)^3 = a^3b^3$ for all $a, b \in G$ and hence $(ba)^2 = a^2b^2$. So $(a^2b^2)^2 = a^4b^4$ and $b^2a^2 = a^2b^2$. Hence in G squares commute.

Now $a^4b^4 = b^4a^4 = (ba)^4$ and so $b^3a^3 = (ab)^3 = a^3b^3$ and hence in G cubes commute. In the solution to Opgave 2003-4B from the UWC it is proved that in this case G is Abelian.

See Chapter 5

Part 2

We define $f_n(x) = x^n$ for $x \in G$. f_m and f_n are homomorphisms.

The case $m = 2$ ($m \leq n$) is trivial because from $(ab)^2 = a^2b^2$ follows immediately $ba = ab$, etcetera.

From $(ab)^m = a^m b^m$ and $(ab)^n = a^n b^n$ ($m < n$) follows $(ba)^{m-1} = a^{m-1} b^{m-1}$ and $(ba)^{n-1} = a^{n-1} b^{n-1}$.

If $k(m-1) = n-1$ and $m-1 = n-m$ we find that $(a^{m-1} b^{m-1})^k = a^{n-1} b^{n-1}$. So $n = 2m - 1$ and $k = 2$ and we may conclude that the $(m-1)$ -th powers commute.

See <https://www.nieuwarchief.nl/serie5/pdf/naw5-2006-07-1-066.pdf>

See also

Vlastimil Dlab, A note on powers of a group, Acta Sci. Math. (Szeged) 25, 1964, pp. 177-178.



Figure 17.1: QR-code link

18

*Problem C NAW 5/6 nr. 4, December 2005**The problem**Introduction*

For a finite affine geometry there are a finite number of points and the axioms are as follows:

1. Given two distinct points, there is exactly one line that includes both points.
2. The parallel postulate: Given a line L and a point P not on L , there exists exactly one line through P that is parallel to L .
3. There exists a set of four points, no three collinear.

We denote the set of points by \mathbf{P} , and the set of lines by \mathbf{L} . We define an automorphism or collineation σ the usual way (a collineation keeps collinearity).

Prove that there exists a point $P \in \mathbf{P}$ with $\sigma(P) = P$ or a line $L \in \mathbf{L}$ with $\sigma(L) = L$ or $\sigma(L) \cap L = \emptyset$.

Solution

Let π be a finite affine plane of order n . π can be canonically embedded in a projective plane $\tilde{\pi}$ of order n by adding a line L_∞ and a point on every line L of π : $L \wedge L_\infty$, where parallel lines L and L' share the same point on L_∞ .

$\bar{\pi}$ has $n^2 + n + 1$ points P_i and the same number of lines L_i . Let $N = n^2 + n + 1$. We define an incidence matrix $A = (a_{ij})$ of order N :

$$a_{ij} = 1 \quad \text{if } P_i \in L_j \quad \text{and} \quad a_{ij} = 0 \quad \text{if } P_i \notin L_j$$

We see that

$$AA^T = A^T A = nI + J \tag{18.1}$$

with J a matrix with every entry 1.

A collineation σ of π can be extended to a collineation of $\bar{\pi}$, also indicated by σ . σ acts on the points P_i as a permutation P and as a permutation Q on the lines L_i . We write P and Q as $(0,1)$ -matrices of order N with entries:

$$\begin{aligned} p_{ij} &= 1 & \text{if } \sigma(P_i) = P_j \\ q_{ij} &= 1 & \text{if } \sigma(L_i) = L_j \end{aligned}$$

and $p_{ij} = 0, q_{ij} = 0$ otherwise.

We now have

$$AQ = PA$$

and according to (1) we have

$$(\det(A))^2 = \det(nI + J) = (n + 1)^2 n^{N-1} > 0$$

So

$$Q = A^{-1}PA$$

P and Q are similar as matrices, but also as permutations. Especially P and Q have the same number of cycles of length one, also called fixed points.

$\sigma(L_\infty) = L_\infty$, so there must be at least one fixed point. If there are no fixed points on L_∞ there is an affine point P with $\sigma(P) = P$. If there is a fixed point on L_∞ , say $L \wedge L_\infty$, then $\sigma(L) \parallel L$, meaning $\sigma(L) = L$ or $\sigma(L) \cap L = \emptyset$.

See <https://www.nieuwarchief.nl/serie5/pdf/naw5-2006-07-2-147.pdf>

Unfortunately, my name was not mentioned by mistake. I think my method is quite original.



Figure 18.1: QR-code link

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Problem B NAW 5/7 nr. 1, March 2006

The problem

Introduction

Let $P = (0,0)$ and $Q = (3,4)$. Find all points $T = (x,y)$ such that

- x and y are integers,
- the lengths of the line segments PT and QT are integers.

Solution

Let the lengths of the line segments PT and QT be denoted by $|PT|$ and $|QT|$. $|PT| - |QT|$ can take integer values ranging from -5 to 5. Let $||PT| - |QT|| = d$ with $d = 0, 1, 2, 3, 4, 5$. So

$$(\sqrt{x^2 + y^2} - \sqrt{(x-3)^2 + (y-4)^2})^2 = d^2$$

and

$$(36 - 4d^2)x^2 + 96xy + (64 - 4d^2)y^2 + (-300 + 12d^2)x + (-400 + 16d^2)y + (d^2 - 25)^2 = 0$$

For $d = 0$ this simplifies to $y = \frac{-6x+25}{8}$ with clearly no integer solutions.

For $d = 5$ the equation simplifies to $4x - 3y = 0$ with solutions $(x, y) = (3k, 4k)$ with integer k .

For $d = 4$ the equation reduces to $-28x^2 + 96xy - 108x - 144y + 81 = 0$ with

$$y = \frac{(2x-9)(14x-9)}{48(2x-3)}$$

clearly with no integer solutions.

For $d = 3$ we get $96xy + 28y^2 - 192x - 256y + 256 = 0$ and

$$x = \frac{(7y - 8)(y - 8)}{24(y - 2)}$$

with obvious solution $(0, 8)$. Less easy to find is $(3, -4)$. There are no other integral solutions.

For $d = 2$ the equation becomes $20x^2 + 96xy + 48y^2 - 252x - 336y + 441 = 0$. Solving for x we get

$$x = \frac{63}{10} - \frac{12}{5}y \pm \frac{1}{5}\sqrt{84y^2 - 336y + 441}$$

Hence $84y^2 - 336y + 441 = 84(y - 2)^2 + 105$ must be a square. Let $Y = y - 2$, then we have to solve Pell's equation $X^2 - 84Y^2 = 105$. This equation has an infinity of solutions based on the fundamental solution $(21, 2)$, but the corresponding x is not an integer. As we can see from

$$x = \frac{63}{10} - \frac{12}{5}(Y + 2) \pm \frac{1}{5}X = \frac{15 - 24Y \pm 2X}{10}$$

For $d = 1$ we have the equation $32x^2 + 96xy + 60y^2 - 288x - 384y + 576 = 0$. Solving for x we get:

$$x = \frac{9}{2} - \frac{3}{2}y \pm \frac{1}{4}\sqrt{6y^2 - 24y + 36}$$

We want $6y^2 - 24y + 36 = 6(y - 2)^2 + 12$ to be a square. Let $Y = y - 2$, then we solve the Pell equation $X^2 - 6Y^2 = 12$. $(X, Y) = (6, 2)$ is a solution. The equation $X^2 - 6Y^2 = 1$ has fundamental solution $(5, 2)$.

We use the following result: If (p, q) is a solution of $x^2 - Dy^2 = N$, and (r, s) is a solution to $x^2 - Dy^2 = 1$, then $x = pr + qsD$, $y = ps + qr$ is also a solution of $x^2 - Dy^2 = N$, because $(pr + qsD)^2 - D(ps + qr)^2 = (p^2 - Dq^2)(r^2 - Ds^2)$.

We define the matrix

$$A = \begin{pmatrix} r & sD \\ s & r \end{pmatrix} = \begin{pmatrix} 5 & 12 \\ 2 & 5 \end{pmatrix}$$

Now we define

$$\begin{pmatrix} X_n \\ Y_n \end{pmatrix} = A^n \begin{pmatrix} p \\ q \end{pmatrix} = A^n \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

For each $n \geq 0$ we define $X = \pm X_n$ and $Y = \pm Y_n$. So for each $n \geq 0$ we find four solutions (x, y) with

$$y = Y + 2 \quad \text{and} \quad x = \frac{X - 6Y + 6}{4}$$

Results from a SAGE [1] program for calculating (x, y) :

(0, 4)	(-3, 4)	(3, 0)	(6, 0)
(-18, 24)	(-45, 24)	(21, -20)	(48, -20)
(-192, 220)	(-459, 220)	(195, -216)	(462, -216)
(-1914, 2160)	(-4557, 2160)	(1917, -2156)	(4560, -2156)
(-18960, 21364)	(-45123, 21364)	(18963, -21360)	(45126, -21360)
(-187698, 211464)	(-446685, 211464)	(187701, -211460)	(446688, -211460)
(-1858032, 2093260)	(-4421739, 2093260)	(1858035, -2093256)	(4421742, -2093256)

Reference

[1] William Stein, David Joyner, SAGE : System for Algebra and Geometry Experimentation, Comm. Computer Algebra 39 (2005) 61-64.

See also: <https://www.nieuwarchief.nl/serie5/pdf/naw5-2006-07-3-219.pdf>

Remark

Those were the early days of SAGE, now SageMath. I was an early adopter of this alternative to the big M's: Maple, Mathematica, Magma, and Mathcad.

SAGE is now called SageMath, see <https://www.sagemath.org/>



Figure 19.1: QR-code link



Figure 19.2: www.sagemath.org

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Problem B NAW 5/7 nr. 2 , December 2005

The problem

Introduction

Imagine a flea circus consisting of n boxes in a row, numbered $1, 2, \dots, n$. In each of the first m boxes there is one flea ($m \leq n$). Each flea can jump upwards or forwards to boxes at a maximum distance $d = n - m$. Each flea has $d + 1$ possible jumps, all equally likely.

The director of the circus has marked m boxes to be special targets. On his sign, all m fleas jump simultaneously.

1. Calculate the probability that after the jump exactly m boxes are occupied.
2. Calculate the probability that all the m marked boxes are occupied.

Solution

Part 1

Let $d = n - m$. The jumps of the fleas correspond to a bipartite graph G . We can associate a $(0,1)$ -matrix B of size $m \times n$ with this graph. We have $b_{ij} = 1$ if and only if $i \leq j \leq i + d$. A matching M of cardinality t corresponds in the matrix B to a set of t ones with no two in the same row. The total number of jumps with exactly m boxes occupied is equal to the number of matchings of size m , which is $\text{per}(B)$, the permanent of B . See [1], p. 44.

The desired probability is $\frac{\text{per}(B)}{(d+1)^m}$.

Part 2

Let $A = \{a_1, a_2, \dots, a_m\}$ be the set of marked boxes, a subset of $\{1, 2, 3, \dots, n\}$, with $1 \leq a_1 < a_2 < \dots < a_m \leq n$ and $(m > 0, m \leq n)$. A successful jump of the fleas can be associated with a permutation of the elements of A . We are looking for permutations π of the elements of A subject to the condition that $k \leq \pi(k) \leq k + d$ for all $1 \leq k \leq m$.

With these restrictions we can associate a $(0, 1)$ -matrix $C = [c_{ij}]$, where $c_{ij} = 1$ if and only if a_j is permitted in position i , meaning $i \leq a_j \leq i + d$.

Compare Problem 29 from NAW 5/3 nr. 1 March 2002.

We define S_C as the set of all permitted permutations. More precisely:

$$S_C = \left\{ \pi \mid \prod_{i=1}^m c_{i\pi(i)} = 1 \right\} \quad (20.1)$$

The number of elements in S_C can be computed by summing over all possible permutations π :

$$|S_C| = \sum_{\pi} \prod_{i=1}^m c_{i\pi(i)} = \text{per}(C) \quad (20.2)$$

where $\text{per}(C)$ is the permanent of C . See [2].

So the desired probability is $\frac{\text{per}(C)}{(d+1)^m}$.



Figure 20.1: Dancing School Problems NAW Dec 2006

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Problem C NAW 5/7 nr. 3, September 2006

The problem

Introduction

Consider the triangle ABC inscribed in an ellipse. For given A the other vertices can be adjusted to maximize the circumference. Prove or disprove that this maximum circumference is independent of the position of A on the ellipse.

Solution

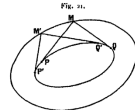
We show that the maximal circumference is independent of point A .

Let E and E' be ellipses with the same foci F_1 and F_2 . Ellipse E' is inside E and sufficiently close to E . Let the points A, B, C and X be on E , such that the line segments AB, BC and CX are tangent to E' . In general the points A and X do not coincide, but if we shrink ellipse E' continuously this will eventually be the case for $E' = E_0$. In this situation we have a triangle ABC inscribed in E with maximal circumference.

If we now move point A along ellipse E , we see that according to a theorem of Poncelet we always have a triangle inscribed in E and circumscribed around E_0 which by construction has a maximal circumference. Chasles, Darboux and others proved that all these triangles have the same maximal circumference. See for example [1] Livre III, Chapitre III, part 176, p. 283. We follow the historical proof of Darboux, using infinitesimal arguments. Let

le triangle intérieur se mouvant.
peut perturber la démonstration à une proposition élégante de Chasles.

Considérons deux ellipses homofocales (E), (E'), et supposons que d'un point quelconque M de l'ellipse extérieure (E), on mène les deux tangentes MP, MQ à l'ellipse intérieure (E'). Nous allons démontrer que, lorsque le point M se déplace sur (E), la



différence entre la somme des tangentes MP, MQ et l'arc PQ de l'ellipse (E')
 $D = MP + MQ - \text{arc } PQ$

demeurera constante.
Pour établir cette proposition, nous emploierons la formule connue

$$dAB = -AA' \cos \angle XAB - BB' \cos \angle XBA,$$

relative à la différentielle d'un segment de droite AB qui, de sa position primitive AB , passe à la position infinitésimale voisine $A'B'$. Si nous l'appliquons successivement aux deux segments MP, MQ en supposant que le point M vienne dans la position voisine M' ,

Figure 21.1: Illustration from note[1]

triangle ABC be defined as above. We have tangents AP and AQ with P and Q on ellipse E_0 . Triangle ABC is a 'billiard triangle', meaning the tangents make equal angles to the normal of E in point A . We move A to A' over an infinitesimal distance and the corresponding points P and Q move to P' and Q' .

Taking into account the properties of tangents we have

$$d(AP) = -AA' \cos(A'AP) + PP'$$

and

$$d(AQ) = -AA' \cos(A'AQ) - QQ'$$

and with the fact that the angles $A'AP$ and $A'AQ$ are supplementary, we get

$$d(AP + AQ) = PP' - QQ' = d(\text{arc}(PQ))$$

Hence the difference $D = (AP + AQ) - \text{arc}(PQ)$ is constant.

Doing this for all vertices of triangle ABC this leads to $3D = O - O'$,

with O being the circumference of ABC and O' the perimeter of the ellipse E_0 .

Conclusion: The maximal circumference is independent of the position of A .

This problem can easily be generalized to an n -sided (convex) polygon inscribed in an ellipse for $n \geq 3$.

See: <https://www.nieuwarchief.nl/serie5/pdf/naw5-2007-08-3-232.pdf>

Remark

For a treatment that is independent of Poncelet's Theorem see George Lion, Variational Aspects of Poncelet's Theorem, Geometricae Dedicata 52, 105-118, 1994.

References

[1] Darboux, G: Principes de Géométrie analytique, Gauthier-Villars, Paris, 1917. Available in facsimile: <https://gallica.bnf.fr>



Figure 21.2: NAW 2006-3



Figure 21.3: NAW September 2007

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*Problem C NAW 5/7 nr. 4, December 2006**The problem**Introduction*

Let G be a finite group of order $p + 1$ with p a prime. Show that p divides the order of $\text{Aut}(G)$ if and only if p is a Mersenne prime, that is, of the form $2^n - 1$, and G is isomorphic to $(\mathbb{Z}/2)^n$.

Solution

Let p be a Mersenne prime with $p = 2^n - 1$ and G be isomorphic to $(\mathbb{Z}_2)^n$, so G is an elementary Abelian group of order 2^n . It is a well-known fact that the group of automorphisms of the elementary Abelian group of order q^r is of order $(q^r - 1)(q^r - q) \dots (q^r - q^{r-1})$, the order of $GL(r, q)$. Hence $p = 2^n - 1$ is a divisor of $|\text{Aut}(G)|$. Let now p be a divisor of $|\text{Aut}(G)|$. $|G| = p + 1$, so there are p elements of G not equal the identity e , say g_1, g_2, \dots, g_p . Clearly $p > 2$, so $p + 1$ is even, so according to the first Sylow Theorem there is a subgroup of G of order 2, and hence there is an element g of order 2. As p is a divisor of the order of the automorphism group of G , we need all possible automorphisms with $g \rightarrow g_i, i = 1, 2, \dots, p$, hence all elements g_i are of order 2.

So G is isomorphic to $(\mathbb{Z}/2)^n$ with $p + 1 = 2^n$ and hence $p = 2^n - 1$ is a Mersenne prime.

See for instance Marshall Hall, Jr. The Theory of Groups Chapter 4.

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*Problem A NAW 5/8 nr. 1, April 2007**The problem**Introduction*

Define the sequence $\{u_n\}$ by $u_1 = 1$, $u_{n+1} = 1 + (n/u_n)$. Prove or disprove that

$$u_n - 1 < \sqrt{n} \leq u_n$$

Solution

By definition we have $u_1 = 1$, $u_{n+1} = 1 + (n/u_n)$ and hence $u_n(u_{n+1} - 1) = n$.

So $\sqrt{n} = \sqrt{u_n(u_{n+1} - 1)}$ is the geometric mean of u_n and $u_{n+1} - 1$ and therefore

$$u_{n+1} - 1 \leq \sqrt{n} \leq u_n$$

for all $n \geq 1$.

Further we have for $n \geq 2$

$$u_n - 1 \leq \sqrt{n-1} < \sqrt{n}$$

This inequality holds also for $n = 1$, so we proved

$$u_n - 1 < \sqrt{n} \leq u_n$$

for all $n \geq 1$. '

24

*Problem B NAW 5/8 nr. 1, April 2007**The problem**Introduction*

Given a non-degenerate tetrahedron (i.e., one whose vertices do not lie in the same plane), what conditions must be satisfied in order that the altitudes intersect at one point?

Solution

Let T be such a tetrahedron with vertices A_0, A_1, A_2 and A_3 in a Euclidean space E . We define $\bar{a}_i = \vec{OA}_i$ and

$$\bar{e}_{ij} = \bar{a}_i - \bar{a}_j \quad (24.1)$$

for $i \neq j$. The vector \bar{e}_{ij} is a direction vector of the edge A_iA_j .

The altitude h_i passing through A_i is determined by the following equations

$$\begin{aligned} \bar{e}_{jk} \cdot (\bar{a}_i - \bar{x}) &= 0 \\ \bar{e}_{jl} \cdot (\bar{a}_i - \bar{x}) &= 0 \\ \bar{e}_{kl} \cdot (\bar{a}_i - \bar{x}) &= 0 \end{aligned} \quad (24.2)$$

with $\{i, j, k, l\} = \{0, 1, 2, 3\}$.

Note that we need only two of them to determine h_i .

We now prove the following lemma:

An altitude h_i intersects with an altitude h_j if and only if

$$\bar{e}_{kl} \cdot \bar{e}_{ij} = 0 \quad (24.3)$$

Proof: From (24.2), the altitude h_i is determined by the equations

$$\begin{aligned} \bar{e}_{jk} \cdot (\bar{a}_i - \bar{x}) &= 0 \\ \bar{e}_{kl} \cdot (\bar{a}_i - \bar{x}) &= 0 \end{aligned}$$

and the altitude h_j is determined by the equations

$$\begin{aligned} \bar{e}_{li} \cdot (\bar{a}_j - \bar{x}) &= 0 \\ \bar{e}_{kl} \cdot (\bar{a}_j - \bar{x}) &= 0 \end{aligned}$$

Let P lie on both h_i and h_j . Let \bar{p} be the point vector of P . Then $\bar{e}_{kl} \cdot (\bar{a}_i - \bar{p}) = 0$ and $\bar{e}_{kl} \cdot (\bar{a}_j - \bar{p}) = 0$ and hence

$$\bar{e}_{kl} \cdot (\bar{a}_i - \bar{a}_j) = \bar{e}_{kl} \cdot \bar{e}_{ij} = 0$$

From (24.3) it follows that $\bar{e}_{kl} \cdot \bar{a}_i = \bar{e}_{kl} \cdot \bar{a}_j$ and so two of the four equations are equal. Three planes intersect in one point unless they are parallel to a line. This is clearly not the case since T is non-degenerate and the vectors \bar{e}_{jk} , \bar{e}_{kl} and \bar{e}_{il} are linearly independent. So h_i and h_j must have a point in common.

By symmetry, the same holds for the altitudes h_k and h_l .

Definition: A tetrahedron is called orthocentric if the altitudes intersect in one point.

Theorem 1. *The following statements are equivalent:*

- i) T is orthocentric.
- ii) All pairs of opposite edges are orthogonal.

Proof:

i) \Rightarrow ii). This follows directly from the lemma.

ii) \Rightarrow i). We now have

$$\bar{e}_{ij} \cdot \bar{e}_{kl} = \bar{e}_{ik} \cdot \bar{e}_{jl} = \bar{e}_{il} \cdot \bar{e}_{jk} = 0$$

So by the lemma, any two altitudes intersect. The four altitudes are not in the same plane, so they must meet at a common point.



Figure 24.1: NAW December 2007

25

*Problem A NAW 5/8 nr. 2, August 2007**The problem**Introduction*

1. Find the largest number c such that all natural numbers n satisfy

$$n\sqrt{2} - \lfloor n\sqrt{2} \rfloor \geq \frac{c}{n}$$

2. For this c , find all natural numbers n such that

$$n\sqrt{2} - \lfloor n\sqrt{2} \rfloor = \frac{c}{n}$$

Solution

Let $p = \lfloor n\sqrt{2} \rfloor$ and $q = n$, so we have to find the largest c for which

$$\sqrt{2} - \frac{p}{q} \geq \frac{c}{q^2} \tag{25.1}$$

holds for all natural numbers q .

We define a function $f : x \rightarrow x^2 - 2$. The equation $f(x) = 0$ has a solution $x = \sqrt{2}$. We note that $\frac{p}{q}$ is an approximation of $\sqrt{2}$ and that $1 \leq \frac{p}{q} < \sqrt{2}$.

Let M be the maximal value of $f'(x) = 2x$ in the interval $[1, \sqrt{2}]$, so $M = 2\sqrt{2}$. Now $f(\frac{p}{q}) = (\frac{p}{q})^2 - 2 = \frac{p^2 - 2q^2}{q^2}$, hence

$$\left| f\left(\frac{p}{q}\right) - f(\sqrt{2}) \right| \leq \frac{1}{q^2}.$$

By the mean-value theorem we get:

$$f\left(\frac{p}{q}\right) - f(\sqrt{2}) = f'(\xi)\left(\frac{p}{q} - \sqrt{2}\right)$$

for some ξ in the interval $[1, \sqrt{2}]$.

Hence

$$\sqrt{2} - \frac{p}{q} = \left| \frac{p}{q} - \sqrt{2} \right| \geq \frac{1}{Mq^2} = \frac{\frac{1}{4}\sqrt{2}}{q^2}$$

Mutatis mutandis we have found $c = \frac{1}{4}\sqrt{2}$.

For this c there clearly is no n satisfying the equality of question 2.

Remark

According to [Hardy]¹ the numbers $\sqrt{5}$ and $2\sqrt{2}$ play a crucial role in approximations of irrational numbers by rationals. For instance the Theorem: Any irrational $\xi \neq \frac{1}{2}(\sqrt{5} - 1)$ has an infinity of rational approximations for which

$$\left| \frac{p}{q} - \xi \right| < \frac{1}{2q^2\sqrt{2}} = \frac{\frac{1}{4}\sqrt{2}}{q^2}$$

Interesting, isn't it?

Reference

[Hardy] Hardy, Wright, An Introduction to the Theory of Numbers, 5th edition, Oxford.

¹ G.H. Hardy and E.M. Wright. *An Introduction to the Theory of Numbers*. Clarendon Press Oxford University Press, Oxford New York, 1979. ISBN 0198531710

26

*Problem C NAW 5/8 nr. 4, December 2007**The problem**Introduction*

Let G be a finite group with n elements. Let c be the number of pairs $(g_1, g_2) \in G \times G$ such that $g_1 g_2 = g_2 g_1$. Show that either G is commutative or that $8c \leq 5n^2$. Show that if $8c = 5n^2$ then 8 divides n .

Solution

We need some elementary group theory and notation. Let $Z(g)$ be the centraliser of $g \in G$ and $K(g)$ the conjugacy class containing g . Let Z denote the centre of G .

We now have $|G| = n$, $c = \sum_{g \in G} |Z(g)|$, and

$$|K(g)| = [G : Z(g)] = \frac{|G|}{|Z(g)|} = \frac{n}{|Z(g)|}$$

We write the ratio

$$\begin{aligned} r &= \frac{c}{n^2} = \frac{\sum_{g \in G} |Z(g)|}{n^2} = \\ &= \frac{1}{n^2} \cdot \sum_{g \in G} \frac{n}{|K(g)|} = \\ &= \frac{1}{n} \cdot \sum_{g \in G} \frac{1}{|K(g)|} = \frac{k}{n} \end{aligned}$$

where k is the number of conjugacy classes.

We note that $r = 1$ if and only if G is commutative. So from now on let G be a non-Abelian group.

We prove the following lemma:

The order of G/Z cannot be a prime number.

Proof: As groups of prime order are cyclic, it suffices to prove that G/Z cannot be cyclic. Suppose G/Z is cyclic, generated by Zx . Then

$$G = Z \cup Zx \cup Zx^2 \cup Zx^3 \cup \dots$$

and arbitrary elements $g_1 = z_1x^i$ and $g_2 = z_2x^j$ clearly commute. This contradicts the assumption that G is non-Abelian.

To maximise the number of conjugacy classes k , we must maximise $|Z|$, the number of conjugacy classes with only one element. From the lemma it follows that $|G/Z| \geq 4$ and so $|Z| \leq \frac{1}{4}|G|$. The other conjugacy classes must have at least two elements. Hence

$$k \leq \frac{1}{4}|G| + \frac{1}{2} \cdot \frac{3}{4}|G| = \frac{5}{8}n$$

so that

$$r \leq \frac{5}{8}$$

and therefore

$$8c \leq 5n^2$$

If $r = \frac{5}{8}$ and hence $k = \frac{5}{8}n$, it follows trivially that 8 divides n .



Figure 26.1: NAW 2007-4 solution

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Problem A NAW 5/9 nr. 1, March 2008

The problem

Introduction

Denote the fractional part of a positive real number x by $\{x\}$.
Evaluate the following double integral:

$$\int_0^1 \int_0^1 \left\{ \frac{x}{y} \right\} \left\{ \frac{y}{x} \right\} dx dy$$

Solution

Let $z = \left\{ \frac{x}{y} \right\} \left\{ \frac{y}{x} \right\}$, then

$$\begin{aligned} z &= \left(\frac{x}{y} - \left\lfloor \frac{x}{y} \right\rfloor \right) \left(\frac{y}{x} - \left\lfloor \frac{y}{x} \right\rfloor \right) = 1 - \frac{x}{y} \left\lfloor \frac{y}{x} \right\rfloor - \frac{y}{x} \left\lfloor \frac{x}{y} \right\rfloor + \left\lfloor \frac{x}{y} \right\rfloor \left\lfloor \frac{y}{x} \right\rfloor \\ &= \begin{cases} 0 & \text{if } x = y \\ 1 - \frac{x}{y} \left\lfloor \frac{y}{x} \right\rfloor & \text{if } x < y \\ 1 - \frac{y}{x} \left\lfloor \frac{x}{y} \right\rfloor & \text{if } x > y \end{cases} \end{aligned}$$

for $0 < x \leq 1$ and $0 < y \leq 1$.

By symmetry we have

$$I = \int_0^1 \int_0^1 z dx dy = 2 \int_0^1 \int_0^y z dx dy = 2 \int_0^1 \int_0^y \left(1 - \left\lfloor \frac{y}{x} \right\rfloor \frac{x}{y} \right) dx dy$$

Now if $n \leq \frac{y}{x} < n + 1$ we have $\left\lfloor \frac{y}{x} \right\rfloor = n$, $z = 1 - n \frac{x}{y}$ and $\frac{1}{n+1}y < x \leq \frac{1}{n}y$.

We define

$$I_n = \int_0^1 \int_{\frac{1}{n+1}y}^{\frac{1}{n}y} (1 - n\frac{x}{y}) dx dy$$

We can easily check that

$$I_n = \frac{1}{4} \left(\frac{1}{n} - \frac{1}{n+1} - \frac{1}{(n+1)^2} \right) = \frac{1}{4n(n+1)^2}$$

and

$$I = 2 \sum_{n=1}^{\infty} I_n = 2 \sum_{n=1}^{\infty} \frac{1}{4n(n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{2n(n+1)^2} = 1 - \frac{\pi^2}{12}$$

28

*Problem B NAW 5/10 nr.2, June 2009**The problem**Introduction*

A magic $n \times n$ matrix of order r is an $n \times n$ matrix whose entries are non-negative integers and whose row and column sums are all equal to r . Let $r > 0$ be an integer. Show that a magic $n \times n$ matrix of order r is the sum of r magic $n \times n$ matrices of order 1.

Solution

Let A be an $n \times n$ magic matrix of order r . We want to assert that A has n positive entries with no two positive entries on a line. This looks trivial but it is not. We need a famous Minimax Theorem of König. This theorem is known in various equivalent forms and there are all kinds of proof in literature. We follow the notation of [1]¹.

Let A be an $m \times n$ matrix with elements from a ring R . The minimal number of lines in A that cover all the non-zero elements of A is equal to the maximal number of non-zero elements in A with no two of the non-zero elements on a line.

We now return to our magic matrix A . If A does not have n positive entries with no two on a line, then by König's theorem we could cover all the positive entries in A with x rows and y columns, where $x + y < n$. All line sums are equal to r , so $(x + y) \cdot r$ counts up to at least nr . We now have $n \leq x + y < n$, which is a contradiction.

¹ Richard Brualdi and Herbert Ryser. *Combinatorial Matrix Theory*. Cambridge University Press, Cambridge England New York, 1991. ISBN 0521322650

Let P_1 be the permutation matrix of order n with ones in the same position as those occupied by the n positive entries of A . Let c_1 be the smallest one of those positive entries. Then clearly $X_1 = A - c_1P_1$ is a magic matrix of order $r - c_1$ with at least one more zero and $A = c_1P_1 + X_1$.

Applying the same argument on X_1 gives $X_2 = X_1 - c_2P_2$, iterating until we get an all zero matrix X_t , we obtain

$$A = c_1P_1 + c_2P_2 + \cdots + c_tP_t$$

Multiplying by J , the $n \times n$ matrix with all ones, we get

$$c_1 + c_2 + \cdots + c_t = r$$

We now note that c_iP_i equals the sum of c_i copies of P_i and P_i is a magic matrix of order 1. So we have decomposed A into r magic matrices of order 1.

References

[1] R.A Brualdi, H.J. Ryser, *Combinatorial Matrix Theory*, Cambridge University Press, 1991.

Remark

See NAW December 2009 for a solution with induction on r using pure Graph Theory.



Figure 28.1: NAW December 2009

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*Problem B NAW 5/10 nr.3, September 2009**The problem**Introduction*Find all functions $f : \mathbf{R}_{>0} \rightarrow \mathbf{R}_{>0}$ such that

$$f(x + y) \geq f(x) + yf(f(x)) \quad (29.1)$$

for all x and y in $\mathbf{R}_{>0}$ *Solution*

Let us rewrite (29.1) slightly:

$$f(x + y) = f(x) + yf(f(x)) + \epsilon(x, y) \quad (29.2)$$

with $\epsilon(x, y) \geq 0$ and $\lim_{y \downarrow 0} \epsilon(x, y) = 0$.

This recalls the lecture notes of Prof. Van der Blij from the early sixties:

$$f(x + h) = f(x) + hf'(x) + \epsilon^*(x, h) \quad (29.3)$$

with $\lim_{h \rightarrow 0} \frac{\epsilon^*(x, h)}{h} = 0$.

We easily see that a solution of the functional-differential equation

$$f'(x) = f(f(x)) \quad (29.4)$$

with domain $D_f \subset \mathbf{R}_{>0}$ can satisfy (29.2).According to references [1] and [2], solutions with $x > 0$ and $f(x) > 0$ exist, but not with $D_f = \mathbf{R}_{>0}$. A typical solution has the following properties:

1. There exists $x_0 > 0$ with $f(x_0) < x_0$
2. There is a σ depending on x_0 with $0 < \sigma \leq 1$ and $f(\sigma) = \sigma$
3. There is $b > \sigma$ with $f(b) < b$
4. There exists $c > b$ with $f(c) = c$
5. The function f is increasing
6. $D_f = \langle 0, c \rangle \subset \mathbf{R}_{>0}$

Such solutions cannot be continued for values of x larger than c .

Our conclusion is that there are no functions f with domain $D_f = \mathbf{R}_{>0}$ satisfying (29.1).

For an elegant solution see: QR-code.



Figure 29.1: QR-code link

References

[1] Eder, Elmar: The functional differential equation $x'(t) = x(x(t))$. J. Diff. Eq. 54 (1984), 390–400.

[2] Wang Ke: On the Equation $x'(t) = f(x(x(t)))$. Funkcialaj Ekvacioj, 33 (1990), 405–425.

Part II

Special Problems

30

*NAW Problem 26**Abstract*

Does there exist a triangle with sides of integral lengths such that its area is equal to the square of the length of one of its sides?

The answer is no.

*The problem**Introduction*

Does there exist a triangle with sides of integral lengths such that its area is equal to the square of the length of one of its sides?

Solution 1.

We can scale to integral sides easily, so suppose we have a triangle with rational sides a , b and c with area A . Then the famous Heron formula gives

$$\begin{aligned}(4A)^2 &= (a+b+c)(a+b-c)(a-b+c)(-a+b+c) \\ &= (x+c)(x-c)(y+c)(-y+c) \\ &= (x^2-c^2)(-y^2+c^2)\end{aligned}$$

with $x = a + b$ and $y = a - b$.



Figure 30.1: NAW December 2001

Without loss of generality we may state that $A = 1$ and $c = 1$. So we have

$$(x^2 - 1)(y^2 - 1) = -16$$

Does this equation have rational solutions?

We make substitutions $X = x$ and $Y = y(x^2 - 1) - x^2$. Rearranging we get

$$(Y + X^2)^2 - (X^2 - 1)^2 = -16(X^2 - 1)$$

Substitution of $X = 1/U - Y$ and solving for U means solving a quartic with discriminant $-2Y^3 - 18Y^2 + 34Y + 306$. So we are looking for a rational solution of

$$Z^2 = -2Y^3 - 18Y^2 + 34Y + 306$$

$Y = -1$ and $Z = 16$ represent a solution, so we can write

$$v^2 = 2u^3 - 12u^2 - 64u + 256$$

where $v = Z$ and $u = -Y - 1$.

This equation of an elliptic curve can be transformed into a Weierstrass equation (N.B.: new meaning of x and y):

$$y^2 - 4xy + 64y = x^3 - 16x^2$$

which can be reduced to minimal form

$$y_1^2 + x_1y_1 + y_1 = x_1^3 - x_1^2 - x_1$$

This means that we have the elliptic curve $(17 A 4 [1,-1,1,-1,0] 0 4)$ from the appropriate Cremona table .

This curve has rank zero, so in the torsion group we find all rational solutions: $(0,0)$, $(0,-1)$ and $(1,-1)$. It is easily verified that this result gives no solutions to our original problem.

Solution 2.

Suppose a triangle ABC exists with integral sides a , b and c with base c , area c^2 and height $CD = 2c$. Let $BD = d$, then

<https://johncremona.github.io/ecdata>



Figure 30.2:
<https://johncremona.github.io/ecdata>

$d^2 = a^2 - 4c^2$ and $d = \sqrt{a^2 - 4c^2}$. We consider the case that ABC is obtuse (The acute case is left as an exercise).

$$\begin{aligned} b^2 &= (c + d)^2 + 4c^2 \\ &= (c + \sqrt{a^2 - 4c^2})^2 + 4c^2 \\ &= a^2 + c^2 + 2c\sqrt{a^2 - 4c^2} \end{aligned}$$

Here a , b and c are integral, so also $d = \sqrt{a^2 - 4c^2}$ must be an integer and therefore the triangles BDC and ADC are Pythagorean.

A well-known result states that a Pythagorean triangle can be parametrised. We leave out some of the details. For BDC we have $a = BC = u^2 + v^2$, $BD = u^2 - v^2$ and $CD = 2uv$, with integers u, v and $u > v$. In triangle ADC we have $AD = s^2 - t^2$ and $CD = 2st$, with integers s, t and $s > t$. While $AB = c = uv$ we have

$$\begin{aligned} s^2 - t^2 &= u^2 - v^2 + uv \\ st &= uv \end{aligned}$$

Dividing the left-hand side of the first equation by st and the right-hand side by uv we get

$$\frac{s}{t} - \frac{t}{s} = \frac{u}{v} - \frac{v}{u} + 1$$

Substitution of $y = \frac{s}{t}$ and $x = \frac{u}{v}$ while $st = uv$ gives

$$y - \frac{1}{y} = x - \frac{1}{x} + 1$$

Multiplying with xy we get

$$x^2y - xy^2 + xy + x - y = 0$$

This cubic can eventually be transformed into a Weierstrass equation of an elliptic curve by Nagell's algorithm. At first we tried this by hand, but the famous `Apecs` lib for Maple V from Ian Connell did it within seconds. The command `Gcub(0, 1, -1, 0, 0, 1, 0, 1, -1, 0, 0, 0)`; returned among others:

$$\text{present curve is, } A17 = [1, -1, 1, -1, 0]$$

Meaning that we have the same elliptic curve as in solution 1.

Connell, `Apecs` (arithmetic of plane elliptic curves), a program written in Maple, available via anonymous ftp from math.mcgill.ca/pub/apecs (1997). This information is obsolete. You can eventually find the `Apecs` file on the Internet Archive Wayback Machine.

Solution 3.

We can scale to integral sides easily, so suppose we have a triangle with rational sides a , b and c with area A . Then the Heron formula gives

$$\begin{aligned}(4A)^2 &= (a+b+c)(a+b-c)(a-b+c)(-a+b+c) \\ &= (x+c)(x-c)(y+c)(-y+c) \\ &= (x^2-c^2)(-y^2+c^2)\end{aligned}$$

with $x = a + b$ and $y = a - b$.

Without loss of generality we may state that $A = 1$ and $c = 1$. So we have

$$(x^2 - 1)(y^2 - 1) = -16$$

Does this equation have rational solutions?

Making the substitutions $U = x$ and $V = y(x^2 - 1) - x^2$ and rearranging we get

$$(V + U^2)^2 - (U^2 - 1)^2 = -16(U^2 - 1)$$

and so

$$2U^2V + 18U^2 + V^2 - 17 = 0$$

with solution $U = 1$ and $V = -1$.

The Apacs command `Gcub(0, 2, 0, 0, 18, 0, 1, 0, 0, -17, 1, -1)`; returned among other information:

present curve is, $A17 = [1, -1, 1, -1, 0]$

Meaning that we have a well-known elliptic curve of rank zero, with torsion group order 4. So the members of the torsion group $(1, -1)$, $(0, 0)$ and $(0, -1)$ are the only rational solutions. It is easily verified that this gives no solutions to our original problem.

Solution 4.

A Heron triangle is a triangle with sides of integral length and integral area. According to K.R.S Sastry [1] every Heron triangle

with sides a, b and c can be parametrised as follows: Let λ be a rational number such that $0 < \lambda \leq 2$.

$$(a, b, c) = (2(m^2 + \lambda^2 n^2), (2 + \lambda)(m^2 - 2\lambda n^2), \lambda(m^2 + 4n^2)),$$

m, n being relatively prime natural numbers with $m > \sqrt{2\lambda}n$. For the sides we have $a \geq c$ and $\gcd(a, b, c) = 1$. The area of this triangle is $\Delta = 2\lambda(2 + \lambda)mn(m^2 - 2\lambda n^2)$.

As one can see, Δ is always a multiple of b . So looking for a solution of our problem, we must consider $\Delta = b^2$, hence

$$2\lambda mn = (2 + \lambda)(m^2 - 2\lambda n^2)$$

Dividing this equation by n^2 gives

$$2\lambda \frac{m}{n} = (2 + \lambda) \left(\left(\frac{m}{n} \right)^2 - 2\lambda \right)$$

Substituting $U = \lambda$ and $V = \frac{m}{n}$ we get

$$UV^2 - 2U^2 - 2UV + 2V^2 - 4U = 0$$

with obvious solution $U = 0, V = 0$.

The ApEcs command `Gcub(0,0,1,0,-2,-2,2,-4,0,0)`; returned:

$$\text{present curve is, } A17 = [1, -1, 1, -1, 0]$$

So again, the same elliptic curve with rank zero. No new rational solutions.

Solution 5.

Suppose we have a triangle ABC with rational sides a, b, c and rational area Δ and $\angle ACB = \theta$. Since $\Delta = \frac{1}{2}ab \sin \theta$ is rational, $\sin \theta$ must be rational.

Using the law of cosines,

$$c^2 = a^2 + b^2 - 2ab \cos \theta$$

So $\cos \theta$ must also be rational. Rational points on the unit circle can be parametrised as:

$$(\cos \theta, \sin \theta) = \left(\frac{1 - t^2}{1 + t^2}, \frac{2t}{1 + t^2} \right)$$

Set $\Delta = 1$ and $c = 1$, so

$$1 = \frac{1}{2}ab \sin \theta \quad \text{and} \quad 1 = a^2 + b^2 - 2ab \cos \theta$$

Hence, $ab = \frac{2}{\sin \theta}$ and

$$a^2 + b^2 = 1 + 2ab \cos \theta = 1 + \frac{2(1 - t^2)}{t}$$

Geometrically, with height $CD = 2$ and $AD = x$, we compute:

$$a^2 + b^2 = (x - 1)^2 + 4 + x^2 + 4 = 2x^2 - 2x + 9$$

Equating gives

$$2x^2t - 2xt + 9t = -2t^2 + t + 2$$

Substituting $U = x$ and $V = t$ gives the cubic:

$$2U^2V - 2UV + 2V^2 + 8V - 2 = 0$$

Using Nagell's algorithm this transforms to minimal form:

$$Y^2 + YX + Y = X^3 - X^2 - 6X - 4$$

This is elliptic curve $(17A2[1, -1, 1, -6, -4], 0, 4)$ with rank zero and torsion group of order 4: $(3, -2), (-1, 0), (-5/4, 1/8)$. No solution to the original problem.

Solution 6.

Suppose a triangle ABC has rational sides a, b, c and area $\Delta = 1$ and $c = 1$. Let $CD = 2$ and $AD = x$. Then:

$$\begin{aligned} a^2 &= (x - 1)^2 + 4 \\ b^2 &= x^2 + 4 \end{aligned}$$

Rational solutions to the second equation are parametrised by

$$x = \frac{4t}{1 - t^2}, \quad b = \frac{2(t^2 + 1)}{1 - t^2}$$

Substituting into the first equation:

$$a^2 = \frac{5t^4 + 8t^3 + 6t^2 - 8t + 5}{(1 - t^2)^2}$$

So we look for rational solutions of

$$y^2 = 5t^4 + 8t^3 + 6t^2 - 8t + 5$$

with solution $(1, 4)$.

Apex command `Quar(5, 8, 6, -8, 5, 1, 4)`; returns:

$$\text{present curve is, } A17 = [1, -1, 1, -1, 0]$$

Again, no new rational solutions.

Solution 7.

Suppose triangle ABC has rational sides a, b, c with $\Delta = 1$ and $c = 1$. Let $CD = 2, BD = x$. The right triangle BDC gives a Pythagorean triangle:

$$x = \frac{u^2 - v^2}{uv}, \quad CD = 2, \quad a = \frac{u^2 + v^2}{uv}$$

Then

$$\begin{aligned} b^2 &= 5 + 2x + x^2 \\ &= \frac{u^4 + 2u^3v + 3u^2v^2 - 2uv^3 + v^4}{u^2v^2} \end{aligned}$$

Set $U = u/v$, then the numerator becomes a quartic:

$$V^2 = U^4 + 2U^3 + 3U^2 - 2U + 1$$

with solution $(0, 1)$. Apex command `Quar(1, 2, 3, -2, 1, 0, 1)`; gives curve $A17 = [1, -1, 1, -1, 0]$. Again, no solution.

Solution 8.

From [1], a parametrisation of Heronian triangles:

$$\begin{aligned} a &= n(m^2 + k^2) \\ b &= m(n^2 + k^2) \\ c &= (m + n)(mn - k^2) \\ \Delta &= kmn(m + n)(mn - k^2) \end{aligned}$$

Assume $\Delta = c^2$, then

$$kmn = (m + n)(mn - k^2)$$

Divide by n^3 :

$$\frac{k}{n} \cdot \frac{m}{n} = \left(\frac{m}{n} + 1\right) \left(\frac{m}{n} - \left(\frac{k}{n}\right)^2\right)$$

Substitute $U = m/n$, $V = k/n$ to get:

$$UV^2 - U^2 + UV + V^2 - U = 0$$

with solution $U = 0$, $V = 0$. Apecs command `Gcub(0,0,1,0,-1,1,1,-1,0,0)`; gives once more curve $A17 = [1, -1, 1, -1, 0]$.

Conclusion.

There is no such triangle.

With thanks to Ian Connell, John Cremona, James Milne and Dave Rusin.

31

NAW Problem 29, January 2003

The Problem

Let n and h be natural numbers with $n > 0$ and A be a subset of $\{1, 2, \dots, n + h\}$ with size n . Count the number of bijective maps $\pi : \{1, 2, \dots, n\} \rightarrow A$ such that $k \leq \pi(k) \leq k + h$ for all $1 \leq k \leq n$.

Solution

Let $A = \{a_1, a_2, \dots, a_n\}$ be a subset of $\{1, 2, 3, \dots, n + h\}$, with $1 \leq a_1 < a_2 < \dots < a_n \leq n + h$ and $(n > 0, h \geq 0)$. We are looking for permutations π of the elements of A with restrictions on permitted positions such that $k \leq \pi(k) \leq k + h$ for all $1 \leq k \leq n$. With this restrictions we can associate a $(0,1)$ -matrix $B = [b_{ij}]$, where $b_{ij} = 1$, if and only if a_j is permitted in position i , meaning $0 \leq a_j - i \leq h$.

We define S_B as the set of all permitted permutations, to be more precise

$$S_B = \left\{ \pi \mid \prod_{i=1}^n b_{i\pi(i)} = 1 \right\}$$

The number of elements of S_B can be calculated by

$$|S_B| = \sum_{\pi} \prod_{i=1}^n b_{i\pi(i)} = \text{per}(B)$$

where $\text{per}(B)$ is the permanent of B .

Example

Let $n = 4$, $h = 3$ and $A = \{2, 3, 5, 6\}$. We can easily see that in this case we have

$$B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

and $\text{per}(B) = 5$, so there are 5 permitted permutations. Being $(2, 3, 5, 6)$, $(3, 2, 5, 6)$, $(2, 3, 6, 5)$, $(3, 2, 6, 5)$ and $(2, 5, 3, 6)$.

Implementation

An implementation of the algorithm can be found on the website of the author:

<https://www.jaapspies.nl/mathfiles/problem29.c>

For a given n and h this program calculates for all possible subsets A the number of allowed bijective maps.

Literature

[1] R.A Brualdi, H.J. Ryser, *Combinatorial Matrix Theory*, Cambridge University Press, 1991.

[2] H. Minc, *Permanents*, Reading, MA: Addison-Wesley, 1978¹.

¹ Henryk Minc. *Permanents*. Addison-Wesley, Reading, Mass., 1978

32

The Dancing School Problems.

We give results related to Problem 29 of the NAW. There are connections to Mathematical Recreation and Graph Theory.

*The Problem**Introduction.*

The Dancing School Problem:

Imagine a group of n ($n > 0$) girls with integer heights ranging from m to $m + n - 1$ cm, and a corresponding group of $n + h$ boys ($h \geq 0$) with heights ranging from m to $m + n + h - 1$ cm. Clearly, m is the minimum height of both boys and girls.

The location is a dancing school. The teacher selects a group of n out of the $n + h$ boys. A girl of height l may now choose a dancing partner from this group of n boys—someone either of her own height or taller, up to a maximum of $l + h$.

How many matchings are possible?

The proof of the equivalence of Problem 29 and the Dancing School Problem is left as an exercise.

A Solution

Let's return to the original problem posed by Lute Kamstra. Let $n > 0$ and $h \geq 0$, and let $A = \{a_1, a_2, \dots, a_n\}$ be a subset of $\{1, 2, \dots, n + h\}$, with $1 \leq a_1 < a_2 < \dots < a_n \leq n + h$. We seek permutations π of the elements of A with restrictions on

allowed positions such that $k \leq \pi(k) \leq k + h$ for all k . With these restrictions, we associate a $(0, 1)$ -matrix $B = [b_{ij}]$, where $b_{ij} = 1$ if and only if a_j is allowed in position i , meaning $i \leq a_j \leq i + h$.

We define S_B as the set of all allowed permutations, more precisely

$$S_B = \left\{ \pi \mid \prod_{i=1}^n b_{i\pi(i)} = 1 \right\} \quad (32.1)$$

The number of elements of S_B is given by

$$|S_B| = \sum_{\pi} \prod_{i=1}^n b_{i\pi(i)} = \text{per}(B) \quad (32.2)$$

where $\text{per}(B)$ is the permanent of B .

For example, let $n = 4$, $h = 3$ and $A = \{2, 3, 5, 6\}$. We can easily see that in this case we have

$$B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

and $\text{per}(B) = 5$, so there are 5 permitted permutations.

Case closed? In general, computing a permanent is a difficult problem with algebraic complexity of order $n^2 2^n$. In some special cases, more efficient algorithms exist.

Some Questions and Answers

Bipartite Graphs

Matrix B can be interpreted as the incidence matrix of a bipartite graph G with vertices in $X = \{1, 2, \dots, n\}$ and $Y = A = \{a_1, a_2, \dots, a_n\}$. An edge of G is a pair (i, a_j) with $b_{ij} = 1$. In our example, the edge set is

$$E = \{(1, 2), (1, 3), (2, 2), (2, 3), (2, 5), (3, 3), (3, 5), (3, 6), (4, 5), (4, 6)\}.$$

A matching in G is a set of disjoint edges. A perfect matching contains n edges.

The number of perfect matchings equals $\text{per}(B)$. Can graph theory be used to compute the number of perfect matchings?

Rook Theory

This is a more algebraic approach to counting $|S_B|$. We interpret the matrix B as an $n \times n$ chessboard. On squares with $b_{ij} = 1$, we may place a rook. Let $r_k(B)$ denote the number of ways to place k non-attacking rooks on the board (i.e., choose k squares such that no two are in the same row or column). This corresponds to matchings with k edges in the associated bipartite graph.

The rook polynomial $r(B, x)$ is defined as

$$r(B, x) = \sum_{k=0}^n r_k(B) x^k$$

Thus, the number of perfect matchings is $r_n(B) = \text{per}(B)$.

Is there a simple way to compute $r(B, x)$ from B ? Probably not—see also the next section.

Configuration Matrix

Let $m = \binom{n+h}{n} = \binom{n+h}{h}$ be the number of subsets X_i of $X = \{1, 2, \dots, n+h\}$. We define a $(0, 1)$ configuration matrix $C = [c_{ij}]$ with $i = 1, \dots, m$, $j = 1, \dots, n+h$, and $c_{ij} = 1$ if and only if $x_j \in X_i$.

The set A from earlier corresponds to the row (0110110) . Can we compute matrix B directly from a row of C ?

Let $A_k = \{a_1, \dots, a_n\}$ be a possible subset of X , corresponding to row k in C . Let h be the number of entries with $c_{kj} = 0$, and n the number with $c_{kj} = 1$. Define matrix $B_k = [b_{ij}]$ of order n , where $b_{ij} = 1$ if and only if $0 \leq a_j - i \leq h$. Then the answer to Problem 29 for A_k is $\text{per}(B_k)$ for $k = 1, 2, \dots, m$.

Related Problems

Dancing School and Rooks

What if the girls take over power and put aside the teacher and they choose directly out of the set of $n+h$ boys (accepting the length restrictions)?

Clearly we can once again associate a bipartite graph G to this problem. The n -set X of girls and the $(n + h)$ -set Y of boys provide the vertices. If a girl a can choose a boy b of appropriate length we have an edge $\{a, b\}$ of G .

The adjacency matrix A has a special form

$$A = \begin{pmatrix} O & B \\ B^T & O \end{pmatrix}$$

Here B is a $(0, 1)$ -matrix of size n by $n + h$ which specifies the adjacencies of the vertices of X and the vertices of Y . We have $b_{ij} = 1$ if and only if $i \leq j \leq i + h$.

A matching M with cardinality n corresponds in the matrix B to a set of n 1's with no two of the 1's in the same row. The total number of matchings with $|M| = n$ is $\text{per}(B)$.

It is clear that our problem can be translated into a Rooks Problem: Find the number of all possible non-attacking placements of n rooks on a $n \times (n + h)$ chessboard, where placing a rook on the i -th row and the j -th column is restricted by the condition $i \leq j \leq i + h$.

Solutions?

Configuration Matrix

We tried to find a recursion from the configuration matrix of the previous section, the so-called direct attack. We define the total number of matchings to be $f(n, h)$. We can rearrange the rows of C such that all rows with $c_{i, n+h} = 1$ are placed together. In this case we have $\pi(n) = n + h$, so the corresponding number of matchings is $f(n - 1, h)$. In all other rows we have $c_{i, n+h} = 0$, counting for $f(n, h - 1)$ matchings, but unfortunately also an extra amount where h comes in! So we can write

$$f(n, h) = f(n - 1, h) + f(n, h - 1) + x(n, h)$$

So far we are not very successful in finding expressions for $x(n, h)$.

In terms of the previous section we may also state

$$f(n, h) = \sum_{k=1}^m \text{per}(B_k)$$

We think this will not lead to anything but trivial solutions, because the calculation of the permanent is a #P-complete problem.

The most effective algorithm in general is Ryser's (see later), which has order of complexity $O(n^2 2^n)$.

Rook Polynomials

In theory, it is possible to calculate the rook polynomial of arbitrary chessboards using the so-called expansion theorem.

Given a chessboard B , let $r_k(B)$ be the number of ways to put k non-attacking rooks on the board, and let

$$r(B, x) = \sum_{k=0}^n r_k(B) x^k$$

be the rook polynomial of board B , and $(r_0(B), r_1(B), \dots, r_n(B))$ the rook vector of B .

We mark a square on board B as special and denote B_s as the chessboard obtained from B by deleting the corresponding row and column. B_d is the board obtained from B by deleting the special square. The ways of placing k non-attacking rooks can now be divided in two cases: those that have the rook in the special square and those that do not. In the first case we have $r_{k-1}(B_s)$ possibilities and in the second $r_k(B_d)$. So we have the relation

$$r_k(B) = r_{k-1}(B_s) + r_k(B_d)$$

This corresponds to

$$r(B, x) = x \cdot r(B_s, x) + r(B_d, x)$$

This is the so-called expansion formula.

Now we can find the rook polynomial of arbitrary boards by repeatedly applying the expansion formula. We think this is only feasible for small sizes, but maybe there are some hidden recursions.

The Permanent to the Rescue

As stated before, we do have a solution to our problem: $per(B)$! B has a clearer form compared to the previous section. So maybe there are solutions lying around.

There is one in Ryser’s Algorithm: Let’s try to translate Theorem 7.1.1 of [1]¹ to our situation. Let $B = [b_{ij}]$ be the $n \times (n + h)$ $(0, 1)$ -matrix with $b_{ij} = 1$ if and only if $i \leq j \leq i + h$. Let r be a number with $h \leq r \leq n + h - 1$ and B_r an $n \times (n + h - r)$ submatrix of B . We define $\prod(B_r)$ to be the product of the row sums of B_r and $\sum \prod(B_r)$ the sum of all such products taken over all choices of B_r . So

$$per(B) = \sum_{k=0}^{n-1} (-1)^k \cdot \binom{h+k}{k} \cdot \sum \prod(B_{h+k})$$

This is a solution, albeit not a very effective one! But maybe we can do better in some cases.

The complements of ...

Intermezzo: Let A be a $(0, 1)$ -matrix with m rows and n columns ($m \leq n$). Let α be a k -subset of the m -set $\{1, 2, \dots, m\}$ and β a k -subset of $\{1, 2, \dots, n\}$. $A[\alpha, \beta]$ is the $k \times k$ submatrix of A determined by rows i with $i \in \alpha$ and columns j with $j \in \beta$.

The permanent $per(A[\alpha, \beta])$ is called a permanental k -minor of A . We define the sum over all possible α and β as

$$p_k(A) = \sum_{\beta} \sum_{\alpha} per(A[\alpha, \beta])$$

We define $p_0(A) = 1$ and note that $p_m(A) = per(A)$. $p_k(A)$ counts the number of k 1’s with no two of the 1’s on the same line, so $p_k(A) = r_k(A)$, the rook vector of A .

According to Theorem 7.2.1 of [1], we can evaluate the permanent of a $(0, 1)$ -matrix in terms of the permanental minors of the complementary matrix $J_{m,n} - A$, where $J_{m,n}$ is the $m \times n$ matrix with all entries 1.

¹ Richard Brualdi and Herbert Ryser. *Combinatorial Matrix Theory*. Cambridge University Press, Cambridge England New York, 1991. ISBN 0521322650

Translated to our matrix B of this section we get

$$\text{per}(B) = \sum_{k=0}^n (-1)^k p_k(J_{n,n+h} - B) \frac{(n+h-k)!}{h!}$$

This is particularly interesting for $h \geq n - 2$, in which case we can easily see that $p_k(J_{n,n+h} - A)$ is independent of h , meaning that $\text{per}(B) = f(n, h)$ is a polynomial in h . For example we have:

$$f(3, h) = h^3 + 3h \quad (h \geq 1),$$

$$f(4, h) = h^4 - 2h^3 + 9h^2 - 8h + 6 \quad (h \geq 2),$$

$$f(5, h) = h^5 - 5h^4 + 25h^3 - 55h^2 + 80h - 46 \quad (h \geq 3),$$

$$f(6, h) = h^6 - 9h^5 + 60h^4 - 225h^3 + 555h^2 - 774h + 484 \quad (h \geq 4),$$

$$f(7, h) = h^7 - 14h^6 + 126h^5 - 700h^4 + 2625h^3 - 6342h^2 + 9072h - 5840 \quad (h \geq 5),$$

We have polynomials up to $f(9, h)$.

Computers are faster now. $n = 12$ or higher maybe feasible. I actually calculated $f(24, h)$ in 18 minutes on my home computer recently (2024).

```
#####
# Copyright (C) 2006 Jaap Spies, jaapspies@gmail.com
#
# Distributed under the terms of the GNU General Public License (GPL):
#
# http://www.gnu.org/licenses/
#####
"""
Usage from sage
sage: attach 'dancing.sage'
sage: dance(4)
h^4 - 2*h^3 + 9*h^2 - 8*h + 6

"""
# use variable 'h' in the polynomial ring over the rationals
h = QQ['h'].gen()

def dance(m):
    """
    Generates the polynomial solutions of the Dancing School Problem
    Based on a modification of theorem 7.2.1 from Brualdi and Ryser,
    Combinatorial Matrix Theory
```

See NAW 5/7 nr. 4 december 2006 p. 285

INPUT: integer m

OUTPUT: polynomial in 'h'

EXAMPLE:

```
sage: dance(4)
      h^4 - 2*h^3 + 9*h^2 - 8*h + 6
```

AUTHOR: Jaap Spies (2006)

"""

```
n = 2*m-2
M = MatrixSpace(QQ, m, n)
A = M([0 for i in range(m*n)])
for i in range(m):
    for j in range(n):
        if i > j or j > i + n - m:
            A[i,j] = 1
rv = A.rook_vector()
s = sum([(-1)^k*rv[k]*falling_factorial(m+h-k, m-k) for k in range(m+1)])
return s
```

The Free Dancing School

What if the girls choose directly from the set of $n + h$ boys and don't accept the length restrictions? They may choose a boy of their own length or taller.

Here again, B is a $(0,1)$ -matrix of size n by $n + h$ which specifies the possible dancing pairs. We now have $b_{ij} = 1$ if and only if $i \leq j \leq n + h$. The number of matchings with cardinality n is $per(B)$.

Let b_1, b_2, \dots, b_m be integers with $0 \leq b_1 \leq b_2 \leq \dots \leq b_m$. The $m \times b_m$ $(0,1)$ -matrix $A = [a_{ij}]$ defined by $a_{ij} = 1$ if and only if $1 \leq j \leq b_i$ (for $i = 1, 2, \dots, m$) is called a Ferrers matrix, denoted by $F(b_1, b_2, \dots, b_m)$. According to [1], we can calculate the permanent with

$$per(F(b_1, b_2, \dots, b_m)) = \prod_{i=1}^m (b_i - i + 1) \quad (32.3)$$

We can associate B with a Ferrers matrix $F(b_1, b_2, \dots, b_n)$ with

$b_i = h + i$. So

$$\text{per}(B) = \prod_{i=1}^n (h + i - i + 1) = (h + 1)^n \quad (32.4)$$

A result we could also have found by direct counting, but we couldn't resist mentioning Ferrers matrices!

References

[1] R.A. Brualdi, H.J. Ryser, *Combinatorial Matrix Theory*, Cambridge University Press.

[2] Nieuw Archief voor de Wiskunde (NAW), Problem Section: Problem 29.

Part III

Permanent Questions

33

Note on the Permanent of a Matrix of order n *The Permanent of a Matrix of order n*

A new algorithm?

The complexity is of the same order as Ryser's algorithm. Multiplication (and addition) with 1 and -1 is extremely easy. So this approach can probably be implemented very efficiently for (0,1)-matrices.

Definitions

Let A be a matrix of order n . The permanent of A is defined by

$$\text{per}(A) = \sum_{\pi} a_{1\pi(1)} a_{2\pi(2)} \cdots a_{n\pi(n)} \quad (33.1)$$

where we sum over all $n!$ possible permutations π of $1, 2, \dots, n$.

We define a vector $\bar{x} = (x_1, x_2, \dots, x_n)^T$ and a vector $\bar{y} = (y_1, y_2, \dots, y_n)^T$. Let $\bar{y} = A\bar{x}$. We define a multivariate polynomial

$$\begin{aligned} P(x_1, x_2, \dots, x_n) &= \prod_{i=1}^n y_i & (33.2) \\ &= (a_{11}x_1 + \dots + a_{1n}x_n) \cdot \\ &\quad (a_{21}x_1 + \dots + a_{2n}x_n) \cdot \\ &\quad \vdots \\ &\quad (a_{n1}x_1 + \dots + a_{nn}x_n) & (33.3) \end{aligned}$$

All terms are of degree n .

In the expansion of (3), we are looking for the coefficient of the term $x_1x_2 \dots x_n$. When we sum over all possible permutations, we get

$$\begin{aligned} & \sum_{\pi} a_{1\pi(1)}x_{\pi(1)} \cdot a_{2\pi(2)}x_{\pi(2)} \cdot \dots \cdot a_{n\pi(n)}x_{\pi(n)} = \\ & = \left(\sum_{\pi} a_{1\pi(1)}a_{2\pi(2)} \dots a_{n\pi(n)} \right) \cdot x_1x_2 \dots x_n \end{aligned}$$

So $per(A)$ is the coefficient of the term $x_1x_2 \dots x_n$.

We define

$$Q(\bar{x}) = \left(\prod_{i=1}^n x_i \right) \cdot P(x_1, x_2, \dots, x_n) \tag{33.4}$$

A Theorem

Now we sum $Q(\bar{x})$ over all possible \bar{x} with $x_i = \pm 1$:

$$\sum_{|\bar{x}|_{\infty}=1} Q(\bar{x}) = \sum_{|x_i|=1} (x_1x_2 \dots x_n) P(x_1, x_2, \dots, x_n)$$

with $|\bar{x}|_{\infty} = 1$ meaning $|x_i| = 1$ for $i = 1, 2, \dots, n$.

As we can easily see, only the term $per(A) \cdot x_1x_2 \dots x_n$ of $P(x_1, x_2, \dots, x_n)$ contributes to this sum, because $x_i^2 = 1$ for $i = 1, 2, \dots, n$. Terms of $Q(x_1, x_2, \dots, x_n)$ with any x_k missing in $P(x_1, x_2, \dots, x_n)$ are counted once as t and once as $-t$, so their total contribution is 0.

There are 2^n possible vectors \bar{x} with $x_i = \pm 1$, so we have proved:

Theorem 1. *The permanent of A is*

$$per(A) = 2^{-n} \cdot \sum_{|\bar{x}|_{\infty}=1} Q(\bar{x}) \tag{33.5}$$

[1] R.A. Brualdi, H.J. Ryser, *Combinatorial Matrix Theory*, Cambridge University Press, 1991.

Remark: This chapter is essentially the note I submitted to the AMM on October 7th, 2003.

34

*The Formula of Spies vs Formula of Glynn**Wikipedia*

While researching for this very book, I stumbled upon a Wikipedia article titled *Computing the Permanent*.

I found the Balasubramanian–Bax–Franklin–Glynn formula. But wait a minute — isn't this my very own formula from 2003? It looks almost the same! It's essentially the same!

My Formula

At the end of 2002, early 2003, I solved Problem 29 of the NAW (see Chapter 31). The solution involved the permanent of a square $(0,1)$ -matrix. For practical calculations, I needed a fast algorithm. One option was to implement Ryser's algorithm, but I discovered a formula that was at least as fast. The derivation was simple, almost elementary. See Chapter 33.

I implemented my algorithm in an arbitrary-precision C/C++ program, which was used to contribute to Neil Sloane's On-line Encyclopedia of Integer Sequences.

Some of the records still stand! Maybe I should give it another try — with today's knowledge and computing power.

A short history from e-mails

In my email of February 11th, 2003, I shared my results with Robbert Fokkink, who was then the editor of the Problem Sec-

https://en.wikipedia.org/Computing_the_permanent



Figure 34.1: [en.wikipedia.org/Computing the Permanent](https://en.wikipedia.org/Computing_the_permanent)

For instance
<https://oeis.org/A087982>,
<https://oeis.org/A088672>,
<https://oeis.org/A089480>,
<https://oeis.org/A089475>,
 and
<https://oeis.org/A089476>

tion of the NAW. In the meantime, I had written a short note including a proof. How ‘new’ was the formula? Robbert suggested sending an email to an expert.

My message to Bruno Codenotti was answered with the suggestion to go straight to the master of permanents: Richard Brualdi. I owned a copy of *Brualdi, Ryser: Combinatorial Matrix Theory*. This book has a chapter on permanents and was the source of the knowledge that led to my solution of Problem 29. Brualdi and Ryser changed my life!

Richard Brualdi wrote: “Your formula is interesting and I cannot recall seeing it before, but maybe it appears buried in some paper or other.” In a follow-up message, he suggested submitting the result as a note to the *American Mathematical Monthly*.

With this recommendation, I sent my note to the AMM on October 7th, 2003. In reply came an email notifying me that my note had been forwarded to Professor William Adkins for refereeing. The refereeing process could take several months, and I would receive a response from Dr. Adkins of Louisiana State University.

No such thing! After more than a year I asked: “What happened to my note? Was it too lightweight and blown away by the wind or what?”

Additionally, I mentioned the implementation of my algorithm in the C programming language, which was faster than an optimized Ryser’s code. I used this software in several contributions to Neil Sloane’s On-Line Encyclopedia of Integer Sequences. (See above.)

From Edwin Clarke: “Thanks for the copy of your note on permanents. It is an interesting approach. It might be a good idea to write it up as a new algorithm and compare it to known algorithms. It looks like the time is about 2^n , which is certainly better than brute-force $n!$.”

In the years that followed, I published some of my work in an article in *Nieuw Archief voor Wiskunde* 5/7 nr. 4 (December 2006): *Dancing School Problems, Permanent Solutions of Problem 29*. In a sidenote, you’ll find my formula for the permanent of

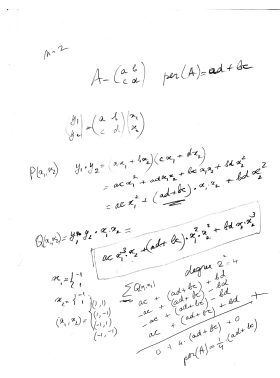


Figure 34.2: Scratch paper

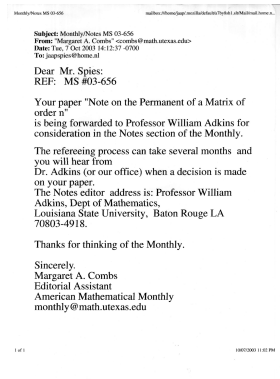


Figure 34.3: Email from the AMM

a square matrix. See also the sidenote on the Van der Waerden Conjecture! It took nearly four years before my solution to Problem 29 was published.

More Recent Developments

As I mentioned in the introduction to this chapter, I encountered Glynn's formula. I was flabbergasted (I like that word!). I wrote an email to my lifeline in mathematics, Robbert Fokkink. Robbert suggested contacting David Glynn in Adelaide. And yes — David replied after reading my long message.

I quote:

“It looks like you could be added to an increasing list of people that have rediscovered this formula for the permanent. (But the first, I think, was Balasubramanian in his PhD thesis in India.) My 2013 paper indeed shows that there are many different formulae related via an algebraic structure called the Veronesean. Maybe I can get someone to add a reference to your articles from 2003–2006 (or others) on the Wikipedia page so it can be noted there.”

Once again, someone suggested I write a short note for the *American Mathematical Monthly*. Again I did. If I may say so, it was a very readable manuscript — with an emphasis on the elementary approach, some history of the permanent, and an educational tone. This time, the rejection came almost immediately.

The motivation:

“In 2010, Glynn published ‘The permanent of a square matrix,’ a five-page note in *The European Journal of Combinatorics* that offered four different formulas for the matrix permanent. Now comes this manuscript, which offers an elementary proof of the first of these. The *Monthly* does publish new, elementary proofs of old results, but only if the old results are famous and/or the new proofs are beautiful and insightful. Unfortunately, neither of these applies here; on the contrary, Glynn's proof was more insightful. Consequently, we must decline your submission for publication.”

I wrote to David Glynn: “So you win 4–1! And your proof was more ‘insightful’. But imagine you have a class of first-year students and you’re teaching some graph theory. You come to the permanent — which method would you prefer to give some insight into its calculation? Do you prefer the insightful approach — ‘connected with invariant theory via the polarization identity for a symmetric tensor’ — or mine, with elementary algebra?”

David Glynn’s first email was also his last. All my follow-up messages disappeared into a black hole.

SageMath

SAGE, now SageMath, was started in 2005 by William Stein. I discovered SAGE at the end of 2005 and became an early adopter. In 2006 and beyond, I contributed quite a bit of code and more.

Sage is Python+, so it is easy to learn and use. SageMath aims to be an open-source alternative to the Big M’s: Magma, Mathematica, Maple, and Matlab. It integrates a lot of other mathematical software.

SageMath has various options for calculating the permanent of matrices over a field. Guess who implemented Ryser’s algorithm?

I wrote implementations of both Glynn’s formula and my own in Sage. In the demo worksheet: www.jaapspies.nl/demoworksheet_perm.pdf, we calculate the permanent of the 3×3 example from Glynn’s 2010 article. Note the differences: in Glynn’s formula, you see column sums in the iterations; in mine, row sums. This is due to the use of right or left multiplication in matrix theory. The results are, of course, the same: the sum of all six diagonal products!

A Sagemath program:

SageMath:
<https://www.sagemath.org>

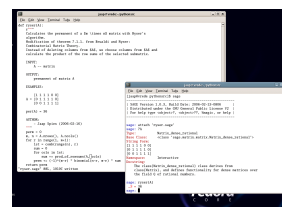


Figure 34.4: Ryser’s Algorithm in SageMath (2006)



Figure 34.5: Demo worksheet

```

R.<a,b,c,d,e,f,g,h,i> = PolynomialRing(ZZ)
A = matrix(R,3,3,[a,b,c,d,e,f,g,h,i])

def cmp(a, b):
    return (a > b) - (a < b)

def demo_perm_spies(M):
    # Thanks to 'xnor' at codegolf.stackexchange.com
    # for the gray code approach
    row_sums = [sum(r) for r in M]
    n = M.dimensions()[0]
    total = 0
    old_gray = 0
    x_i = +1
    powers_of_2_dict = {2**i:i for i in range(n)}
    num_loops = 2**(n-1)
    for num in range(1, num_loops + 1):
        prod = 1
        print("iteration:",num, "rowsums: ",row_sums)
        for x in row_sums:
            prod *= x
        print("prod of row_sums:",prod)
        total += x_i * prod
        print("\ntotal:", total)
        new_gray = num ^^ (num >> 1)
        diff = old_gray ^^ new_gray
        diff_index = powers_of_2_dict[diff]
        new_vector = M.column(diff_index)
        dir = 2 * cmp(old_gray,new_gray)
        for i in range(n):
            row_sums[i] += new_vector[i] * dir
        x_i = -x_i
        old_gray = new_gray
    return total / num_loops

demo_perm_spies(A)

[snipped]

total: 4*c*e*g + 4*b*f*g + 4*c*d*h + 4*a*f*h + 4*b*d*i + 4*a*e*i
c*e*g + b*f*g + c*d*h + a*f*h + b*d*i + a*e*i

```

And now?

In the NAW nr. 1 March 2020 there is an article "A formula for the permanent". <https://www.nieuwarchief.nl/serie5/pdf/naw5-2020-21-1-027.pdf>

In the Dutch Wikipedia page on Permanents we now see the Formula of Spies. [https://nl.wikipedia.org/wiki/Permanent_\(wiskunde\)](https://nl.wikipedia.org/wiki/Permanent_(wiskunde))

I still hope there will be an addition to the English version of the Wikipedia page! Volunteers?



Figure 34.6: NAW: A formula for the permanent



Figure 34.7: nl.wikipedia.org Permanent_(wiskunde)

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